FORMULATION OF THE HOMOLOGICAL FUNCTORS OF SOME BIEBERBACH GROUPS WITH DIHEDRAL POINT GROUP

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FAKULTI SAINS & MATEMATIK
UNIVERSITI PENDIDIKAN SULTAN IDRIS

2015

ABSTRACT

The purpose of this research is to produce the formulas of the homological functors of some Bieberbach groups of dimension five with dihedral point group of order eight. The homological functors consist of the nonabelian tensor square, G-trivial subgroup of the nonabelian tensor square, the central subgroup of the nonabelian tensor square, the nonabelian exterior square and Schur multiplier. The computational method for polycyclic groups is used to determine the formulas of the functors. Selected theorems, lemmas and definitions are also used in the computation. The findings of this research are the new formulas of the homological functors and also the generalization of one of the functors that is the central subgroup of the nonabelian tensor square. As a conclusion, the nonabelian tensor square and the nonabelian exterior square are found to be not abelian. While G-trivial subgroup of the nonabelian tensor square, the central subgroup of the nonabelian tensor square and Schur multiplier are abelian. The implication of the findings enables the properties of other Bieberbach groups to be explored. In addition, the results can be beneficial not only to mathematicians in terms of theories and applications but also to chemists and physicists.

FORMULASI FUNGTOR HOMOLOGI BEBERAPA KUMPULAN BIEBERBACH DENGAN KUMPULAN TITIK DWIHEDRON

ABSTRAK

Kajian ini bertujuan menghasilkan formula fungtor homologi bagi beberapa kumpulan Bieberbach berdimensi lima dengan kumpulan titik dwihedron berperingkat lapan. Fungtor homologi terdiri daripada kuasa dua tensor tak abelan, sub-kumpulan G-remeh bagi kuasa dua tensor tak abelan, sub-kumpulan pusat bagi kuasa dua tensor tak abelan, kuasa dua peluaran tak abelan dan pekali Schur. Kaedah kumpulan polikitaran digunakan untuk menghasilkan formula-formula fungtor tersebut. Teorem, lemma dan definisi terpilih juga digunakan di dalam pengiraan. Dapatan kajian ini ialah formula-formula baharu fungtor homologi dan juga pengitlakan satu dari fungtor tersebut iaitu sub-kumpulan pusat bagi kuasa dua tensor tak abelan. Kesimpulannya, kuasa dua tensor tak abelan dan kuasa dua peluaran tak abelan adalah didapati tidak abelan. Manakala sub-kumpulan G-remeh bagi kuasa dua tensor tak abelan, sub-kumpulan pusat bagi kuasa dua tensor tak abelan dan pekali Schur adalah abelan. Implikasi dapatan kajian ini membolehkan sifat-sifat kumpulan Bieberbach yang lain diterokai. Di samping itu, ia boleh memberi manfaat bukan sahaja kepada ahli matematik dari segi teori dan aplikasi tetapi juga kepada ahli kimia dan fizik.

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G^{ab}	- Abelianization of G, G/G'
$Z(G)$	- Center of the group G
$[g,h]$	- Commutator of g and h
h^g	- Conjugate of h by g
G'	- Derived subgroup of G
$G \times H$	- Direct product of G and H
$\in$	- Element of
$\notin$	- Not element of
$G \wedge G$	- Exterior square of G
F_n^{ab}	- Free abelian group of rank n
$G \cong H$	- G is isomorphic to H
$\langle X \vert R \rangle$	- Group presented by generators X and relators R
$\langle x \rangle$	- Group generated by the element x
$H \leq G$	- H is a subgroup of G
$H \lhd G$	- H is normal in G
$B_i(j)$	- i^{th} Bieberbach group with dihedral point group of order 8 with dimension j
C_n	- Cyclic group of order n
$G \otimes G$	- Nonabelian tensor square of G
$\nabla(G)$	- The central subgroup of the nonabelian tensor square
$M(G)$	- Schur multiplier of G
$G \tilde{\otimes} G$	- Symmetric square of G
D_8	- Dihedral point group of order 8

gH, Hg	-	Left coset, and right coset of H, respectively, with coset representative g
G/H	-	Quotient group of G by H

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Bieberbach groups are torsion free crystallographic groups. These groups are extensions of a finite point group P and a free abelian group L of finite rank. Then there is a short exact sequence

$$1 \to L \to G \to P \to 1$$

such that $G/L \cong P$. Here, L is called a lattice group.

The properties of the Bieberbach groups can be explored by computing the homological functors of the groups. The homological functors of a group such as $G \otimes G$, the nonabelian tensor square, $J(G)$, G-trivial subgroup of $G \otimes G$, $\nabla(G)$, the central subgroup of $G \otimes G$, $G \wedge G$, the nonabelian exterior square, $M(G)$, Schur Multiplier, $\Delta(G)$, the subgroup of $J(G)$, $G \tilde{\otimes} G$, symmetrix square and $\tilde{J}(G)$, the kernel of the $G \tilde{\otimes} G$. Homological functors of various groups have been discovered by many researchers throughout the years.

The nonabelian tensor square of the group, which was introduced by Brown and Loday (1987) is vital in the determination of other homological functors of a group. It is an interesting group theoretic construction which arises from applications in homotopy theory of a generalized Van Kampen theorem. The studies of the nonabelian tensor square, $G \otimes G$ of certain groups reveal many properties of these groups. In 1999, Kappe, Nor Haniza and Visscher determined the nonabelian tensor square of 2-generator 2-group of class 2 by classifying this group into four types. The nonabelian tensor square of all types of this group was determined. Recently, Nor Haniza (2012) studied the nonabelian tensor squares and the homological functors for infinite nonabelian 2-generator groups of nilpotency class two.

Besides that, the homological functors of the groups are related to each other. $J(G)$ can be obtained by computing the kernel of the homomorphism $\kappa : G \otimes G \to G'$. $G \wedge G$ is the factor group of $G \otimes G/\nabla(G)$ and $M(G)$ is isomorphic to the kernel of the homomorphism $\kappa : G \wedge G \to G'$. Those functors are also important to explore the properties of the groups.

1.2 Research Background

A Bieberbach group is defined as a torsion free crystallographic group. Some researches related to the computation of the homological functors of some Bieberbach groups had been done since 2009 starting with Rohaidah (2009), Nor'ashiqin (2011), and then followed by Hazzirah Izzati, Nor Haniza, Nor Muhainiah, Rohaidah and Nor'ashiqin (2013). Rohaidah (2009) started the study related to the homological functors of Bieberbach groups where she computed the nonabelian tensor squares of certain Bieberbach groups with cyclic point groups. Her works had been extended by Hazzirah Izzati et al. (2013) with the computation of the Schur multipliers of certain Bieberbach groups with abelian point groups.

Nor'ashiqin (2011) computed the nonabelian tensor square of the centerless Bieberbach group of dimension four with dihedral point group of order eight. In 2011, Hazzirah Izzati, Nor Haniza, Nor Muhainiah and Mohd Sham conducted a research on computing the nonabelian tensor squares and some homological functors of all 2-Engel groups of order at most 16. In this research, the homological functors of some Bieberbach groups with dihedral point group of order eight will be investigated.

1.3 Problem Statement

Nor'ashiqin (2011) found that there are 73 centerless Bieberbach group with dihedral point group of order eight using the Crystallographic Algorithms and Tables (CARAT) package. One of them has dimension four, eleven of them have dimension five and the rest of them have dimension six. Nor'ashiqin (2011) focused on a

Bieberbach group of dimension four. While in this research, the Bieberbach groups of dimension five are chosen for the investigation.

In this research, the main focus is to compute the homological functors of some Bieberbach groups with dihedral point group of order eight and come out with a new formulation of the computation of the homological functors of these groups. In this research, $G \otimes G$, $J(G)$, $\nabla(G)$, $G \wedge G$ and $M(G)$ are computed for the Bieberbach groups with point group D_8 of dimension five. A generalization of the homological functors of some Bieberbach groups with dihedral point group will be constructed. The results in Nor'ashiqin (2011) will be used as a guide in order to compute the nonabelian tensor squares and the other homological functors of some Bieberbach groups of dimension five with dihedral point group of order eight. Group, Algorithms and Programming (GAP) software is used to assist in the computation of the homological functors in this research.

1.4 Research Objectives

The objectives of this research are:

(i) to compute the nonabelian tensor squares of some Bieberbach groups of dimension five with dihedral point group of order eight,

(ii) to construct a formula of the computation of the other homological functors such as $J(G)$, G-trivial subgroup of $G \otimes G$, $\nabla(G)$, the central subgroup of $G \otimes G$, $G \wedge G$, the nonabelian exterior square and $M(G)$, the Schur Multiplier of some Bieberbach groups of dimension five with dihedral point group of order eight,

(iii) to develop the generalization of $\nabla(G)$ of a Bieberbach group with dihedral point group of order eight up to dimension n.

1.5 Scope of the Study

In this research, the group considered will be some Bieberbach groups of dimension five with dihedral point group of order eight. In generalizing the homological functors, the group is limited to a Bieberbach group with dihedral point group of order eight up to dimension n.

1.6 Significance of the Study

The major contribution of this research will be the new theoretical results on computing the homological functors of Bieberbach groups with dihedral point group of order eight. The properties of the Bieberbach groups of dimension five with dihedral point group of order eight and computing their homological functors will contributes as a foundation in determining the generalization of the homological functors of Bieberbach groups of other dimension with dihedral point group of order more than eight. Thus, this research provides new results in Group Theory. These results obtained can be used for further research in related areas.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter describes the researches that had been done by other researchers on the Bieberbach groups and the homological functors of some groups. The method of computing the nonabelian tensor square of a group is briefly discussed and some background of CARAT and GAP software package used in this research are explained.

2.2 Some Basic Definitions and Concepts on Homological Functors

The homological functors of a group were originated in homotopy theory as well as in algebraic K-theory. One of the most important homological functors is non-abelian tensor square, which will be first computed in this research. It is important since it is vital in determination of the other homological functors of a group G, as we can see in the definitions of other homological functors given below. The nonabelian tensor square is a special case of the nonabelian tensor product where the product is defined if the two groups act on each other in a compatible way and their action are taken to be conjugation.

The definitions of the homological functors considered in this research are given in the following.

Definition 2.1 (Brown & Loday, 1987) $G \otimes G$

The nonabelian tensor square $G \otimes G$ of a group G is generated by the symbols $g \otimes h$ for all $g, h \in G$, subject to relations

$$gh \otimes k = (g^h \otimes k^h)(h \otimes k) \text{ and } g \otimes hk = (g \otimes k)(g^k \otimes h^k)$$

for all $g, h, k \in G$ where $g^h = h^{-1}gh$ denotes the conjugate of g by h.

Definition 2.2 (Ellis, 1998) $J(G)$

The G-trivial subgroup of $G \otimes G$, denoted as $J(G)$, is defined by the kernel of the homomorphism κ where $\kappa : G \otimes G \rightarrow G'$ with $\kappa(g \otimes h) = [g, h]$.

Definition 2.3 (Brown & Loday, 1987) $\nabla(G)$

The central subgroup of $G \otimes G$, denoted as $\nabla(G)$, is a subgroup of $J(G)$ is defined by $\nabla(G) = \{g \otimes g | g \in G\}$.

Definition 2.4 (Ellis, 1998) The Nonabelian Exterior Square

The exterior square of G is defined as $G \wedge G = (G \otimes G)/\nabla(G)$.

Definition 2.5 (Ellis, 1998) Schur Multiplier

The Schur Multiplier of G is defined as $M(G) = J(G)/\nabla(G)$.

2.3 Bieberbach Groups

One of the special kinds of crystallographic groups is called a Bieberbach group. Bieberbach group is known as a crystallographic group which has no elements of finite order is described in the following proposition:

Proposition 2.1 (Hiller, 1986)

A crystallographic group G is a Bieberbach group if and only if it is torsion-free.

Gahler (1997) had done a research on computing with crystallographic groups by using software known as CrystGap. The algorithms in CrystGap were build to compute with crystallographic groups of arbitrary dimension, in particular for the determination of the Wyckoff positions and maximal subgroups of a crystallographic group besides for the determination of all crystallographic group types for a given point group. These algorithms had been implemented in the computer algebra system GAP.

Cid and Schulz (2001) focused on the computation and the classification of five and six dimensional torsion-free crystallographic groups. In their research, the basis of an algorithm that decides torsion-freeness of a crystallographic group as well as the triviality of its centre is described clearly.

Malfait and Szczepanski (2003) conducted a research on the structure of the (outer) automorphism group of a Bieberbach group. Malfait and Szczepanski developed a necessary and sufficient condition to decide whether the normalizer of a finite group of integral matrices is polycyclic-by-finite or is containing a nonabelian free group. The result concludes the (outer) automorphism group of a Bieberbach group is either polycyclic-by-finite or has a non-cyclic free subgroup.

2.4 The Homological Functors of Groups

In this section, some previous researches on the homological functors of groups are discussed. In 2004, Blyth, Morse and Redden computed the nonabelian tensor square for the free 2-Engel group of rank $n > 3$. The nonabelian tensor square for one of the group's finite homomorphic image, namely, the Burnside group of rank n and exponent three also had been computed. In their research, the computations were performed using method developed in Ellis and Leonard (1995).

Rohaidah (2009) was the first researcher who studied about the nonabelian tensor square of Bieberbach group where she computed the nonabelian tensor squares of Bieberbach groups with cyclic point group. Her works had been extended by Hazzirah Izzati et al. (2013) where they computed the Schur multipliers of certain Bieberbach groups with abelian point groups.

Nor'ashiqin (2011) computed the nonabelian tensor square of the centerless Bieberbach group of dimension four with the dihedral point group of order eight. By using the method developed by Blyth and Morse (2009), the result showed that the polycyclic groups can give the generators and presentation of the nonabelian tensor square of the group that computer computational cannot give.

Hazzirah Izzati et al. (2011) conducted a research on computing the nonabelian tensor squares and some homological functors of all 2-Engel groups of order at most 16. The Groups, Algorithms and Programming (GAP) software was used to get some patterns for the generalization of some homological functors.

The main goal of computing the nonabelian tensor square is to construct the presentation of $G \otimes G$ in terms of its generators. The structure of $G \otimes G$ in terms of its central extensions has been investigated by Brown and Loday (1987). Their findings showed that the group $\nabla(G) = \{g \otimes g | g \in G\}$ is a central subgroup of $G \otimes G$. The nonabelian exterior square, $G \wedge G$ is a quotient group $(G \otimes G)/\nabla(G)$ (Blyth & Morse, 2009). Here $J(G)$ denotes the kernel of the mapping $G \otimes G \to G' : g \otimes h \mapsto [g, h]$ and $\Gamma(G/G')$ is the Whitehead quadratic functor (Ellis, 1998). While the Schur multiplier, $M(G)$ is isomorphic to the kernel of $G \wedge G \to G' : g \wedge h \mapsto [g, h]$ (Brown, Johnson & Robertson, 1987). These relations with well-known constructions suggest the interests in the computation of $G \otimes G$. All the homological functors discussed above are decribed in the following commutative diagram of the homological functors.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J(G) & \longrightarrow & G\otimes G & \xrightarrow{\kappa} & G' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \tilde{J}(G) & \longrightarrow & G\tilde{\otimes}G & \xrightarrow{\kappa''} & G' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M(G) & \longrightarrow & G\wedge G & \xrightarrow{\kappa'} & G' & \longrightarrow & 1
\end{array}$$

Figure 2.1: Commutative diagram of homological functors

2.5 Computing the Nonabelian Tensor Square of a Group

There are some methods used in computing the nonabelian tensor square $G \otimes G$. One of them is its definition as given in Definition 2.1. For finite group, Brown et al. (1987) used this definition to compute $G \otimes G$ by forming the finite presentation. They also used the definition to compute the nonabelian tensor squares for all groups G of order at most 30.

For infinite group, the crossed pairing method is always used to compute the nonabelian tensor square which is abelian, since the definition will lead to the infinite presentation. By using this method, a mapping $\Phi : G \times G \to L$ determined, where G and L are groups. The crossed pairing method determines a unique homomorphism $\Phi^* : G \otimes G \to L$. Kappe et al. (1999) used this method to determine the nonabelian tensor square of 2-generator 2-groups of class two. This method had also been used by Bacon (1994) to compute the nonabelian tensor square for the free nilpotent group of class two of finite rank. However, it is difficult to verify that any mapping is a crossed pairing when the nonabelian tensor square is not abelian.

Rocco (1991) conducted a research on a construction related to the nonabelian tensor square of a group. The method developed by him always used by other researchers to compute the nonabelian tensor square of group where in his method, he introduced the group $\nu(G)$ as given in the following definition.

Definition 2.6 (Rocco, 1991)

Let G be a group with presentation $\langle G|R\rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi : g \rightarrow g^{\varphi}$ for all $g \in G$. The group $\nu(G)$ is defined to be

$$\nu(G) = \langle G, G^{\varphi}|R, R^{\varphi}, {}^{x}[g, h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x^{\varphi}}[g, h^{\varphi}], \text{ for all } x, g, h \in G\rangle.$$

Ellis and Leonard (1995) have shown that $[G, G^{\varphi}]$ which is the subgroup of $\nu(G)$ that is described in Proposition 2.3, page 19, is isomorphic to the nonabelian tensor square of the group G as given in the following theorem.

Theorem 2.1 (Ellis & Leonard, 1995)

Let G be a group. The map $\sigma : G \otimes G \rightarrow [G, G^{\varphi}] \lhd \nu(G)$ defined by $\sigma(g \otimes h) = [g, h^{\varphi}]$ for all g, h in G is an isomorphism.

Due to the limitations of the usage of crossed pairing method, Blyth and Morse (2009) extend the method involving a cover group $\nu(G)$ to compute the nonabelian tensor square for infinite group or in particular the polycyclic group. They proved that if G is a polycyclic group, then $G \otimes G$ is polycyclic. So, $\nu(G)$ is a polycyclic too as shown in the following proposition.

Proposition 2.2 (Blyth & Morse, 2009)

If G is polycyclic, then $\nu(G)$ is polycyclic.

Blyth and Morse (2009) showed that the nonabelian tensor square of group can be computed using computer method and by hand, aided with the commutator calculus listed in the section of preliminary results. In this research, the nonabelian tensor squares of the some Bieberbach groups of dimension five with dihedral point group of order eight are computed by using the method introduced by Blyth and Morse (2009). This method will be elaborated more in the preliminary results.

2.6 GAP and CARAT

Group, Algorithms, and Programming (GAP) is a system for computational discrete algebra, which emphasis on computational group theory. GAP provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects. GAP is used in research and teaching for studying groups and their representations, rings, vector spaces, algebras, combinatorial structures, and more. The system, including source, is distributed freely.

In this research, GAP is used to assist the hand calculation of the homological functors of the some Bieberbach groups of dimension five with dihedral point group of order eight. GAP is also used to help researcher to get some pattern in order to construct the generalization of the homological functors of the Bieberbach group with dihedral point group of order eight up to dimension n.

Crystallographic AlgoRithms And Tables (CARAT) is a computer package, which provides GAP interface routines to some of the stand-alone programs of CARAT. It provides routines for the computation of normalizers and conjuga-