Mixed finite element methods for nonlinear equations: a priori and a posteriori error estimates

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ABSTRACT

A priori error estimation provides information about the asymptotic behavior of the approximate solution and information on convergence rates of the problem. Contrarily, a posteriori error estimation derives the estimation of the exact error by employing the approximate solution and provides a practical accurate error estimation. Additionally, a posteriori error estimates can be used to steer adaptive schemes, that is to decide the refinement processes, namely local mesh refinement or local order refinement schemes. Adaptive schemes of finite element methods for numerical solutions of partial differential equations are considered standard tools in science and engineering to achieve better accuracy with minimum degrees of freedom.

In this thesis, we focus on a posteriori error estimations of mixed finite element methods for nonlinear time dependent partial differential equations. Mixed finite element methods are methods which are based on mixed formulations of the problem. In a mixed formulation, the derivative of the solution is introduced as a separate dependent variable in a different finite element space than the solution itself. We implement the H^1-Galerkin mixed finite element method (H1MFEM) to approximate the solution and its derivative. Two nonlinear time dependent partial differential equations are considered in this thesis, namely the Benjamin-Bona-Mahony (BBM) equation and Burgers equation. Our a posteriori error estimations are based on implicit schemes of a posteriori error estimations, where the error estimators are locally computed on each element. We propose a posteriori error estimates by using the approximate solution produced by H1MFEM and use the a posteriori error estimates to compute the local error estimators, respectively for the BBM and Burgers equations. Then, we prove that the introduced a posteriori error estimates are accurate and efficient estimations of the exact errors.

The last part of this study is on numerical studies of adaptive mesh refinement schemes for the two equations mentioned above. By implementing the introduced a posteriori error estimates, we propose adaptive mesh refinement schemes of H1MFEM for both equations.

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Chapter 1

Introduction

1.1 Subject of study

Disciplines such as engineering, physics, economics and biology involve many real models which are translated into solving mathematical models, e.g. partial differential equations in space and time. One of the well-known mathematical models is the boundary value problem (BVP). For example, a description of waves in electromagnetic and fluid dynamics is represented by a wave equation with specified boundary conditions, which is often stated as a boundary value problem.

Consistent with the application of the BVP in real problems, the number of numerical methods and analysis for solving the BVP is rapidly growing. In general, there is no closed form for the exact solution u of the BVP. Numerical methods as the finite difference method, the finite element method, the finite volume method, and spline interpolation are used as a tool to compute the approximate solution U_h of the exact solution u. Motivated by this situation, our study is considering the finite element method (FEM) for the boundary value problems.

During the approximation of the BVP, it is normal to question *"How good are the approximate solutions produced by the numerical methods? When should we stop the computation process and which of the approximate solutions should be taken as the best approximation to the real problem?"* In order to answer these questions, one way is to perform another approximation in a specified norm of the exact error e,

$$\|e\|_H = \|u - U_h\|_H ,$$

i.e. by letting the exact error be approximated such as

$$\|E\|_H \approx \|e\|_H .$$

Extra concern should be put on deciding the methods to compute values of E, the error estimator. The computability and cost of computation are factors that are considered in deciding the efficiency of the error estimation.

In this study, we focus on a posteriori error estimation which is a method to compute the error estimator. Details about the a posteriori error estimation of finite element methods can be found in [6, 8] and the references therein. The a posteriori error estimation is based on a situation where we have the approximate solutions U_h which are generated by a FEM, then our aim is to obtain a quantitative estimate for the exact error e measured in a specified norm.

A posteriori error estimates provide useful indications of the accuracy of a calculation and provide a basis for adaptive mesh refinement schemes. We will give the details of a posteriori error estimation in Chapter 3.

1.2 Scope of study

This study focuses on a posteriori error estimation of a mixed finite element method (mixed FEM) for the Benjamin-Bona-Mahony equation and Burgers equation. The mixed FEM is a FEM which is based on a mixed formulation of the problem. In a mixed formulation, the derivative of the solution u is introduced as a separate dependent variable in a different finite element space than the solution itself. In this study, we implement the H^1-Galerkin mixed finite element method which is based on the procedure introduced by Pani [45]. Our scope of study can be categorized into three main parts.

The first part is devoted to a posteriori error estimation of the H^1-Galerkin mixed finite element method for the Benjamin-Bona-Mahony equation. The Benjamin-Bona-Mahony equation is a nonlinear equation which is widely used in modelling physical problems involving long waves. The Benjamin-Bona-Mahony equation is studied by Benjamin et al. as an alternative to the Korteweg-de Vries equation for describing unidirectional long dispersive waves [12].

Secondly, we study a posteriori error estimation of the H^1-Galerkin mixed finite element method for the Burgers equation. The Burgers equation is a well-known equation and named after Johannes Martinus Burgers [18, 19]. The Burgers equation is also known as a nonlinear diffusion equation, and a simplified version of Navier-Stokes equation. We will give details about the H^1-Galerkin mixed finite element method, the Benjamin-Bona-Mahony equation and the Burgers equation in the following chapter (Chapter 2).

The last part of this study is on adaptive schemes for two equations mentioned above. The a posteriori error estimates are known as a fundamental component in the designation of efficient adaptive algorithms for solving partial differential equations. By implementing the a posteriori error estimates introduced in the first two objectives, our third objective is on numerical studies of adaptive schemes for the Benjamin-Bona-Mahony equation and the Burgers equation.

1.3 Structure of thesis

This thesis consists of six chapters. Chapter 1 is the introduction. In Chapter 2, some important function spaces, theorems and results are reviewed. We complete Chapter 2 with an introduction for H^1-Galerkin mixed finite element method, the Benjamin-Bona-Mahony equation and the Burgers equation.

Chapter 3 is devoted to a general framework of a posteriori error estimation. In this chapter, we present some known a posteriori error estimation techniques and the procedure of a posteriori error estimation considered in this study.

In Chapter 4, we present the first contribution of the thesis which is a posteriori error estimation of H^1-Galerkin mixed finite element method for the Benjamin-Bona-Mahony equation. In this chapter, we propose some error estimators to compute the error estimation of the Benjamin-Bona-Mahony equation. We prove that the introduced a posteriori error estimates are accurate and efficient approximations of the exact errors.

We finish this chapter with some numerical experiments.

Our second contribution (Chapter 5) is a posteriori error estimation of H^1-Galerkin mixed finite element method for the Burgers equation. This chapter consists of analysis and numerical studies of a posteriori error estimation of H^1-Galerkin mixed finite element method for the Burgers equation.

Our third contribution (Chapter 6) is numerical studies of adaptive schemes for the Benjamin-Bona-Mahony and Burgers equations. We present the procedure of the adaptive schemes for both equations where the approximate solutions are computed by H^1-Galerkin mixed finite element method and the a posteriori error estimations are proposed in Chapter 4 and Chapter 5. We finish the chapter with numerical experiments.

Chapter 2

Preliminaries

This chapter provides a range of fundamental results which will be used in the remainder of the thesis. We begin by introducing some important function spaces. We then introduce the fundamental results for variational formulation of differential equations. Important theorems and results that will be used in the analysis of finite element methods are introduced in the next section. We finish this chapter with an introduction on finite element discretization, the H^1-Galerkin mixed finite element method, the Benjamin-Bona-Mahony equation and the Burgers equation, respectively in Section 2.5, Section 2.6 and Section 2.7.

2.1 Function spaces

All of the results stated in this section are well-known and can be found in different literatures; see e.g. [16, 53]. Since the equations we study in this thesis are posed in one spatial dimension, we mention only results for this case.

We let $\Omega = (a, b)$ be an open subset in $\mathbb{R}$ and u be a scalar function defined on Ω. For $p \in [1, \infty]$, the Lebesgue space $L^p(\Omega)$ is defined as

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} \big| \, \|u\|_{L^p(\Omega)} < \infty\}.$$

The $L^p(\Omega)$-norm is defined by

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p},$$

for $0 < p < \infty$, and

$$\|u\|_{L^\infty(\Omega)} := \inf\{C \geq 0 : |u(x)| \leq C \quad \text{for almost all} \quad x \in \Omega\}$$

for $p = \infty$. A special role is taken when $p = 2$. The $L^2(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)\,dx \quad u, v \in L^2(\Omega)$$

and the norm $\|u\|_{L^2(\Omega)}$.

Let $\mathbb{C}^m(\Omega)$ be the space of all functions $\phi : \Omega \longrightarrow \mathbb{R}$ such that $\phi, \phi', \cdots, \phi^{(m)}$ are continuous on Ω. The space $\mathbb{C}_0^m(\Omega)$ denoted the space of all functions $\phi \in \mathbb{C}^m$ such that $\phi(x) = 0$ for all $x \in \Omega_0$ for some bounded subset Ω_0 of Ω.

We recall the definition of derivative in a weak sense. A function $u \in L^p(\Omega)$ is called the weak derivative of order $m = 1, 2, 3, \ldots$ of a function $v \in L^p(\Omega)$ if

$$\int_\Omega u(x)\phi(x)\,dx = (-1)^m \int_\Omega v(x)\phi^{(m)}(x)\,dx \quad \forall\phi \in \mathbb{C}_0^m(\Omega).$$

In the following part, we recall the Sobolev spaces and norms to be used in this thesis. The Sobolev space $W_p^k(\Omega)$, $1 \leq p < \infty$ and $k = 1, 2, \ldots$ is defined as

$$W_p^k(\Omega) := \{u \in L^p(\Omega) : u', u'', \cdots, u^{(k)} \text{exist in the weak sense}\}$$

and $W_p^k(\Omega)$ norm is defined by

$$\|u\|_{W_p^k(\Omega)} := \left(\sum_{i=0}^{k} \left\|u^{(i)}\right\|_{L^p(\Omega)}^p\right)^{1/p}.$$

When $p = 2$, we have $W_2^k(\Omega) = H^k(\Omega)$, which is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_{H^k(\Omega)} := \int_\Omega \left(uv + u'v' + \cdots + u^{(k)}v^{(k)}\right)\,dx \quad \forall u, v \in H^k(\Omega),$$

and norm

$$\|u\|_{H^k(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + \|u'\|_{L^2(\Omega)}^2 + \cdots + \left\|u^{(k)}\right\|_{L^2(\Omega)}^2\right)^{1/2} \qquad \forall u \in H^k(\Omega).$$

The space $H_0^k(\Omega)$ contains all functions in $H^k(\Omega)$ whose traces are zero at a and b.

In the case $p = \infty$, $W_\infty^k(\Omega)$ norm is defined by

$$\|u\|_{W_\infty^k(\Omega)} := \max_{0 \leq i \leq k} \left\|u^{(i)}\right\|_{L^\infty(\Omega)}.$$

2.2 Notations

In the remaining part of the thesis, we use the following notations for the norm spaces and inner products. For any $p \in [1, \infty]$ and any normed vector space D, by $L^p(D)$ we denote the space $L^p(0, T; D)$ of all functions defined in $[0, T]$, with values in D. We will write $\|\cdot\|_{L^p(L^\infty)}$ and $\|\cdot\|_{L^p(H^1)}$ instead of $\|\cdot\|_{L^p(L^\infty(D))}$ and $\|\cdot\|_{L^p(H^1(D))}$. We will also write $H^0(D) = L^2(D)$. The $H^n(D)$ norm, for $n = 0, 1, \dots$ is represented by $\|u(t)\|_n$ instead of $\|u(t)\|_{H^n(D)}$. Similarly, we will write $\|u\|_{W^1_\infty(H^n)}$ instead of $\|u\|_{W^1_\infty(0,T;H^n(D))}$.

In general, the inner product in $H^s(X)$ is denoted by $\langle \cdot, \cdot \rangle_{H^s(X)}$, where $s = 0, 1, \dots$ and X is a subset of $\mathbb{R}$. In particular, when $s = 0$ and $X = \Omega$ we write $\langle \cdot, \cdot \rangle_0$ instead of $\langle \cdot, \cdot \rangle_{H^0(\Omega)}$. When $s = 1$ and $X = \Omega$, we write $\langle \cdot, \cdot \rangle_1$ instead of $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$.

Besides that, when there is no confusions we omit the dependence of the function on t to avoid crowded notations. For example, we write $\langle u, v \rangle_s$ instead of $\langle u(t), v(t) \rangle_{H^s(\Omega)}$. Finally, for $l > 0$, we define the local inner product in $H^s(\Omega_l)$ by

$$\langle u, v \rangle_{s,\Omega_l} = \int_{\Omega_l} u(x)v(x)dx \quad \forall u, v \in H^s(\Omega_l). \tag{2.2.1}$$

2.3 Important theorems and results

The following well-known results will be frequently used. They are recalled here for the reader's convenience.

Theorem 2.3.1 (Imbedding Theorem [16, Theorem 1.4.6]). *Let k be a positive integer and p be a real number in the range $1 \le p < \infty$ such that*

$$k \ge 1 \quad \textit{when} \quad p = 1$$

$$k > 1/p \quad \textit{when} \quad p > 1.$$

Then there is a constant C such that for all $u \in W^k_p(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \le C \|u\|_{W^k_p(\Omega)} .$$

Lemma 2.3.2 (Gronwall's Lemma [11, Theorem 4.2] or [28]). *Let φ, ψ and θ be locally integrable functions defined on $[0, T]$ which satisfy*

$$\theta(t) \ge 0 \quad \textit{and} \quad \varphi(t) \le \psi(t) + \int_0^t \theta(s)\varphi(s)\, ds \qquad \forall t \in [0, T].$$

Then

$$\varphi(t) \leq \psi(t) + \int_0^t \theta(s)\psi(s) \exp\left[\int_s^t \theta(r)dr\right] ds \quad \forall t \in [0, T].$$

If ψ is a constant, then

$$\varphi(t) \leq \psi \exp\left[\int_0^t \theta(s)\, ds\right].$$

Lemma 2.3.3 (General Gronwall's Lemma [11, Theorem 4.3] or [13, Section 3]). *If β is a positive constant and θ is a non-decreasing function satisfying $\theta(s) > 0$ for $s > 0$, then the inequality*

$$\varphi(t) \leq \beta + \int_0^t \theta[\varphi(\tau)]\, d\tau \quad \forall t \in [0, T]$$

implies

$$\varphi(t) \leq \Theta^{-1}(t) \quad \forall t \in [0, T^*]$$

where Θ^{-1} is the inverse of

$$\Theta(\sigma) = \int_\beta^\sigma \frac{ds}{\theta(s)}, \quad \sigma \geq 0,$$

and $T^* = \min(T, T_1)$ with $[0, T_1]$ being the range of Θ.

Proof. Let

$$\psi(t) = \beta + \int_0^t \theta[\varphi(\tau)]\, d\tau.$$

Then from $\varphi(t) \leq \psi(t)$ and the monotonicity of θ we deduce

$$\frac{\psi'(t)}{\theta[\psi(t)]} = \frac{\theta[\varphi(t)]}{\theta[\psi(t)]} \leq 1.$$

This implies

$$\frac{d}{dt}\Theta[\psi(t)] \leq 1.$$

By integrating from 0 to t and noting that $\Theta[\psi(0)] = 0$, we obtain

$$\Theta[\psi(t)] \leq t.$$

Now if $t \in [0, T^*]$ then by applying the inverse function Θ^{-1}, we obtain $\psi(t) \leq \Theta^{-1}(t)$, and thus the required inequality follows from $\varphi(t) \leq \psi(t)$. □

2.4 Finite element discretization

In this section, we introduce the hierarchical basis functions of finite element spaces used in the thesis. We partition the interval $\Omega = (a, b)$ into

$$a = x_1 < x_2 < \cdots < x_{N+1} = b, \tag{2.4.1}$$

and let $h_l := x_{l+1} - x_l$, $l = 1, \ldots, N$, and $h := \max_l h_l$. We define the linear basis functions by using the hat functions ϕ_{l1} on (x_{l-1}, x_{l+1}) for $l = 2, \ldots, N$, i.e.,

$$\phi_{l1}(x) = \begin{cases} \frac{x - x_{l-1}}{h_{l-1}}, & x_{l-1} \le x < x_l, \\ \frac{x_{l+1} - x}{h_l}, & x_l \le x < x_{l+1}, \\ 0, & \text{otherwise.} \end{cases}$$

At the endpoints of Ω we define

$$\phi_{11}(x) = \begin{cases} \frac{x_2 - x}{h_1}, & x_1 \le x < x_2, \\ 0, & \text{otherwise,} \end{cases} \tag{2.4.2}$$

and

$$\phi_{N+1,1}(x) = \begin{cases} \frac{x - x_N}{h_N}, & x_N \le x < x_{N+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.3}$$

For $l = 1, \ldots, N$ and $k = 2, 3, 4, \ldots$, functions ϕ_{lk} are defined as antiderivatives of the Legendre polynomials P_{k-1} of degree $k - 1$ scaled to the subinterval $[x_l, x_{l+1}]$, i.e.,

$$\phi_{lk}(x) = \begin{cases} \frac{\sqrt{2(2k-1)}}{h_l} \int_{x_l}^{x} P_{k-1}(y)\, dy, & x_l \le x < x_{l+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.4}$$

Figure 2.4 shows functions $\phi_{l,k}$ of degree $k = 2, \ldots, 5$ on the reference element $[-1, 1]$.

Let $\mathcal{S}_h$ be the space of piecewise linear functions on Ω i.e.,

$$\mathcal{S}_h := \text{span}\,\{\phi_{11}, \phi_{21} \ldots, \phi_{N+1,1}\},$$

and $\mathring{\mathcal{S}}_h$ its subspace consisting of functions vanishing at a and b, i.e.,

$$\mathring{\mathcal{S}}_h := \text{span}\,\{\phi_{21}, \ldots, \phi_{N,1}\}.$$

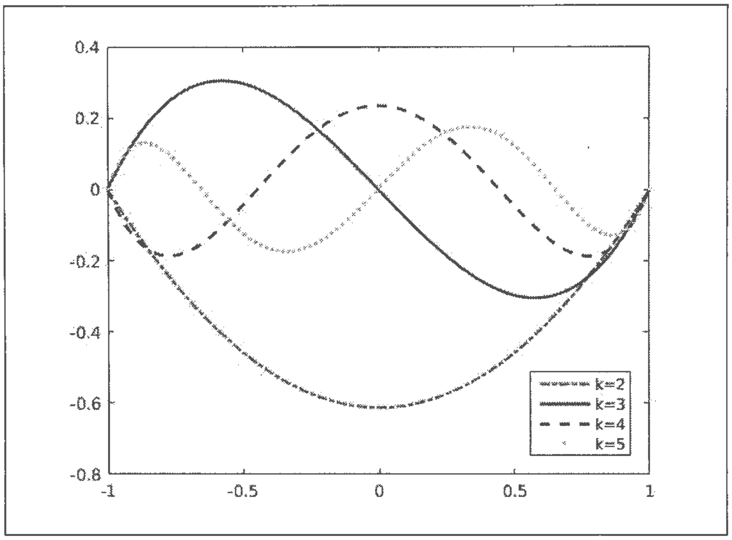

Figure 2.1: Hierarchical shape functions of degrees 2 (−·), 3 (solid), 4 (−−) and 5 (··) on reference element $[-1, 1]$.

The spaces of bubble functions in Ω are defined by

$$\mathcal{S}_h^k := \operatorname{span}\{\phi_{1k}, \dots, \phi_{Nk}\}, \qquad k \geq 2,$$

where, for $l = 1, \dots, N$ and $k = 2, 3, 4, \dots$, ϕ_{lk} is defined by (2.4.4).

For $p \in \mathbb{N}$ and $p \geq 2$, let $\mathcal{V}_h^p$ and $\mathring{\mathcal{V}}_h^p$ be finite dimensional subspaces of $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively, defined by

$$\mathcal{V}_h^p := \mathcal{S}_h + \sum_{k=2}^{p} \mathcal{S}_h^k, \quad \text{and} \quad \mathring{\mathcal{V}}_h^p := \mathring{\mathcal{S}}_h + \sum_{k=2}^{p} \mathcal{S}_h^k. \tag{2.4.5}$$

With $\chi_h \in \mathring{\mathcal{V}}_h^p$ and $w_h \in \mathcal{V}_h^p$, we have the following approximation properties

$$\inf_{\chi_h \in \mathring{\mathcal{V}}_h^p} \left\{ \|u - \chi_h\|_0 + h \|\partial_x(u - \chi_h)\|_0 \right\} \leq Ch^{p+1} \|u\|_{p+1} \quad \forall u \in H_0^1(\Omega) \cap H^{p+1}(\Omega) \tag{2.4.6}$$

and

$$\inf_{w_h \in \mathcal{V}_h^p} \left\{ \|v - w_h\|_0 + h \|\partial_x(v - w_h)\|_0 \right\} \leq Ch^{p+1} \|v\|_{p+1} \quad \forall v \in H^{p+1}(\Omega). \tag{2.4.7}$$

2.5 H^1-Galerkin mixed finite element method

The mathematical analysis and applications of mixed FEM have been widely developed since decades ago. For example, a general analysis for this kind of methods is studied by Brezzi [17]. A mixed FEM is a type of FEM which is based on a mixed formulation of the problem. The mixed FEM is originally considered for problems where there are possibilities of having numerical ill-posedness if discretized by using the normal FEM. An example of such problems is computation of stress and strain fields in an almost incompressible elastic body. Besides that, the mixed FEM is also applied for cases where we have to discretize the gradient of the solution. The need to approximate the gradient of the solution is originated from solid mechanics problems which require more accurate approximations of certain derivatives of the displacement [16]. The mathematical elements of classical mixed FEM can be found in the books on mathematical theory of FEM [16, 53].

By using mixed FEM, the original problem is reformulated into a problem of two bilinear forms and two finite element spaces. As an example we consider the following one dimensional parabolic partial differential equation:

$$\partial_t u(x,t) - \partial_{xx} u(x,t) = f(x,t), \quad x \in \Omega = (0,1), \quad t \in (0,T], \quad T < \infty, \tag{2.5.1}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in [0,T], \tag{2.5.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.5.3}$$

By using a mixed formulation, the derivative of the solution u is introduced as a second unknown. The second order problem is reformulated into a system of first order equations having the form

$$v(x,t) = \partial_x u(x,t), \tag{2.5.4}$$

$$\partial_t u(x,t) - \partial_x v(x,t) = f(x,t) \tag{2.5.5}$$

with boundary condition (2.5.2) and initial condition (2.5.3). We note that, with $\alpha \in H^1(\Omega)$ and $\beta \in H^0(\Omega)$ are arbitrary test functions, the solution $(u, v) \in H^0(\Omega) \times H^1(\Omega)$ also solves the weak formulation

$$\langle v(t), \alpha \rangle_0 = - \langle u(t), \partial_x \alpha \rangle_0 \quad \forall \alpha \in H^1(\Omega) \tag{2.5.6}$$

$$\langle \partial_t u(t), \beta \rangle_0 - \langle \partial_x v(t), \beta \rangle_0 = \langle f(t), \beta \rangle_0 \quad \forall \beta \in H^0(\Omega). \tag{2.5.7}$$

It should be noted that the boundary condition $u = 0$ (see (2.5.2)) is implicitly contained in the formulation (2.5.6)–(2.5.7). Using integration by parts in (2.5.6), we have

$$\langle v, \alpha \rangle_0 = - \langle u, \partial_x \alpha \rangle_0 = \langle \partial_x u, \alpha \rangle_0 \quad \forall \alpha \in H^1(\Omega),$$

and hence, formally, $v = \partial_x u$ in Ω and $u = 0$ at the endpoints of Ω. Since $v = \partial_x u$ from (2.5.6), noting that $\partial_x v \in H^0(\Omega)$ and taking $\beta = \partial_t u - \partial_x v - f \in H^0(\Omega)$ in (2.5.7), we have (2.5.1).

This way of mixed formulation needs two finite dimensional spaces $\mathcal{W} \subset H^0(\Omega)$ and $\mathcal{V} \subset H^1(\Omega)$ which are required to satisfy inf-sup condition or Ladyzhenskaya-Babuška-Brezzi (LBB) condition to have a stable numerical scheme. Details of the mixed formulation by classical mixed FEM for a general parabolic partial differential equation can be found in [30].

In this study, we implement a mixed FEM called H^1-Galerkin mixed finite element method (H1MFEM) which is based on an approach suggested by Pani et. al for nonlinear parabolic equations and second order hyperbolic equations [45, 46]. The H1MFEM is closely related to least square mixed methods in that the second order partial differential equation is reformulated into a system of first order partial differential equations with a new unknown defined as the flux. Studies on the least square mixed finite element method can be found in [20, 21, 49, 48] and the references therein. By using the H1MFEM, a problem is reformulated into a system of first order partial differential equations, which allows the approximation for u and its gradient v.

As an example, we consider parabolic partial differential equation (2.5.1)–(2.5.3). Using the H1MFEM, equation (2.5.1) is reduced to a system of first order equations by defining a new variable $v = \partial_x u$. As a consequence, we have (2.5.4)–(2.5.5). By multiplying (2.5.4) by $\partial_x \alpha$ and (2.5.5) by $-\partial_x \beta$ where $\alpha \in H_0^1(\Omega)$ and $\beta \in H^1(\Omega)$ we have

$$\langle v, \partial_x \alpha \rangle_0 = \langle \partial_x u, \partial_x \alpha \rangle_0 \quad \forall \alpha \in H_0^1(\Omega) \tag{2.5.8}$$

and

$$\langle \partial_t v, \beta \rangle_0 + \langle \partial_x v, \partial_x \beta \rangle_0 = -\langle f, \partial_x \beta \rangle_0 \quad \forall \beta \in H^1(\Omega). \tag{2.5.9}$$

For the first term in (2.5.9), we have used integration by parts and the Dirichlet boundary conditions $\partial_t u(0,t) = \partial_t u(1,t) = 0$.

The weak formulation by H1MFEM is formulated as: Given $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, find $(u, v) : [0, T] \to H_0^1(\Omega) \times H^1(\Omega)$, satisfying for $t > 0$

$$\langle v(t), \partial_x \alpha \rangle_0 = \langle \partial_x u(t), \partial_x \alpha \rangle_0 \quad \forall \alpha \in H_0^1(\Omega) \tag{2.5.10}$$

$$\langle \partial_t v(t), \beta \rangle_0 + \langle \partial_x v(t), \partial_x \beta \rangle_0 = -\langle f(t), \partial_x \beta \rangle_0 \quad \forall \beta \in H^1(\Omega) \tag{2.5.11}$$

and for $t = 0$,

$$\langle v(0), \beta \rangle_0 = \langle \partial_x u_0, \beta \rangle_0 \quad \forall \beta \in H^1(\Omega). \tag{2.5.12}$$

If $u \in W_\infty^1(0, T; H_0^1(\Omega) \cap H^2(\Omega))$, $v \in W_\infty^1(0, T; H^1(\Omega))$ and (u, v) satisfies (2.5.10)–(2.5.11) then (u, v) satisfies (2.5.4)–(2.5.5). Indeed, by using integration by parts we deduce from (2.5.10) that $\partial_x(v - \partial_x u) = 0 \in W_\infty^1(0, T; H^0(\Omega))$, which implies

$$v(x,t) = \partial_x u(x,t) + g(t) \quad \text{a.e. in} \quad \Omega \times (0, T) \tag{2.5.13}$$

for some function g depending on t. We note that we also have

$$v(x, 0) = \partial_x u(x, 0) + g(0).$$

By integrating over Ω, noting (2.5.12), we infer $g(0) = 0$. On the other hand, it follows from (2.5.13) and (2.5.11) (with $\beta = 1$) that

$$\int_\Omega \partial_{tx} u + g'(t) = 0, \tag{2.5.14}$$

implying $g'(t) = 0$. Hence $g \equiv 0$, i.e. (u, v) satisfies (2.5.4). This immediately gives (2.5.5).

Some of the attractive features of the H1MFEM are firstly this method does not require the LBB condition. Secondly, finite element spaces of u and v are allowed to be of different polynomial degrees. For example, by using the H1MFEM, the approximate solutions U_h and V_h of the finite element spaces $\mathring{\mathcal{V}}_h^p$ and $\mathcal{V}_h^q$ (see (2.4.5)) are allowed to be of different polynomial degrees, i.e. we can have different values of p and q where $p, q \geq 1$. Thirdly, this procedure required extra regularity of the solution which gives a better order of convergence for v, in $H^0(\Omega)$ norm [45]. For one dimensional cases, the orders of convergence obtained by H1MFEM are

$$\|u - U_h\|_1 \leq Ch^{\min(p,q+1)} \tag{2.5.15}$$

and

$$\|v - V_h\|_1 \leq Ch^{\min(p+1,q)}, \tag{2.5.16}$$

which are comparable with results generated by a classical mixed FEM. Details of the mixed formulation by the H1MFEM for a general parabolic partial differential equation can be found in [45].

In 2007, the H1MFEM is adapted for a priori error estimation of the Burgers equation [47]. Besides that, Tripathy et. al studied on the superconvergence properties of the H1MFEM for second order elliptic equations [56]. Recently, Zhang et. al studied the H1MFEM with the linearised Crank-Nicolson for couple BBM equations [59].

In this thesis, we are interested in a posteriori error estimations of the H1MFEM for the BBM and Burgers equations. Mixed finite element methods allow approximation to the solution of the BBM and Burgers equations and its derivative, by reformulating the BBM and Burgers equations into a system of first order equations. Therefore, instead of dealing with second order nonlinear partial differential equations, the problem is reformulated and the computation is less hard compared to the approximation by using a normal finite element method. Mixed finite element methods give better orders of convergence for the unknown derivative by requiring extra regularity of the unknown. To the best of our knowledge, this is the first time the procedure of a posteriori error estimation in this study (to be explained in Chapter 3) is applied to the BBM and Burgers equations, where the approximate solutions are computed by using the H1MFEM.

2.6 The Benjamin-Bona-Mahony equation

The Benjamin-Bona-Mahony (BBM) equation

$$\partial_t u(x,t) - \partial_{xxt} u(x,t) + u(x,t)\partial_x u(x,t) + \partial_x u(x,t) = 0, \tag{2.6.1}$$

where $\partial_t = \partial/\partial t$, $\partial_{xxt} = \partial^3/\partial x^2 \partial t$ and $\partial_x = \partial/\partial x$ is studied by Benjamin et al., with $u(x,t)$ being considered in a class of real nonperiodic functions defined for $-\infty < x < \infty$ and $t \geq 0$ [12]. The BBM equation is studied in flows of fluid. Examples where the BBM equation is used are acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma and acoustic waves in anharmonic crystal.

The BBM equation is studied as an alternative and improvement of the Korteweg-de Vries (KdV) equation

$$\partial_t u(x,t) + \partial_{xxx} u(x,t) + u(x,t)\partial_x u(x,t) + \partial_x u(x,t) = 0, \tag{2.6.2}$$

particularly for describing unidirectional long dispersive waves. In general, the KdV model in physical science and engineering has difficulty with the dispersion ratio; a ratio of dispersion's effect in a medium, when a wave is travelling within the medium. The dispersion term $\partial_{xxx} u$ in the KdV model has a tendency to emphasise the short-wave components which is unnatural with respect to the original physical problem. The dispersion relation $\partial_{xxt} u$ in the BBM model overcomes this difficulty by giving a bounded dispersion relation [37]. Besides that, modelling with the BBM equation also overcomes the stability problem with high wave number components in the KdV model.

Details on the uniqueness and stability of the BBM model for long waves in nonlinear dispersive systems can be found in [12]. The existence and uniqueness of (2.6.1) and its non-homogeneous form are studied by Benjamin et al. Besides that, the uniqueness, global existence and continuous dependence of solutions on initial and boundary data for model equation (2.6.1) with an additional term $-\partial_{xx} u$ are studied by Bona and Dougalis [14]. Another general case of the BBM equation, namely

$$\partial_t u(x,t) - \partial_{xxt} u(x,t) + \partial_x f(u) = g(x,t) \tag{2.6.3}$$

where $f \in C^1(\mathbb{R})$ and $g \in L^\infty(0,T;L^2(0,1))$, is studied by Medeiros and Miranda [39]. They prove existence, uniqueness and regularity of (2.6.3). The BBM equation is also studied for periodic solutions (periodic with respect to the x variable) [23], [38]. For higher dimensions, a study on the existence, uniqueness and regularity is conducted by Goldstein et. al [27].

Since decades ago, initial boundary value problems for various generalized BBM equations have been studied. For example, in [32], a linearised method which is based on a differential quadrature method is studied as a new method to approximate the BBM equation on a semi-infinite interval. A linearised Crank-Nicolson H1MFEM is studied for coupled BBM equations in [59]. Besides that, a numerical study on the BBM equation with a mixed FEM (differently from the method studied in Chapter 4 of this thesis) can be found in [33]. In this study, we are interested on a posteriori error estimation for the BBM equation, where the approximate solutions are computed by using the H1MFEM.

2.7 The Burgers equation

The Burgers equation

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = \nu \partial_{xx} u(x,t) \tag{2.7.1}$$

is a fundamental one dimensional nonlinear partial differential equation occurring in various areas of mathematical modelling, particularly in mathematical models of turbulence and shock wave theory. Solution $u(x,t)$ can be considered as a quantity of a velocity for space x and time t. The value of ν is a small parameter known as a viscosity coefficient of the fluid motion, which is related to the Reynolds number $R = \frac{1}{\nu}$. The Burgers model has been studied as the simplest form of nonlinear advection term $u\partial_x u$ and dissipation term $\nu\partial_{xx}u$ for simulating the physical phenomena of wave motions.

Since decades ago, the Burgers model became an interest of researchers due to the tendency of a steep gradient (shocks) which almost becomes discontinuous when the viscosity coefficient $\nu = 0$ in (2.7.1) i.e.

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0. \tag{2.7.2}$$

Equation (2.7.2) is also known as inviscid Burgers equation.

Because of the nonlinear convection term and the occurrence of the viscosity term, the Burgers equation (2.7.1) shows similar features with Navier-Stokes equation and it is viewed as the simplified version of the Navier-Stokes equation. Due to the complexity in obtaining the analytical solutions, many researchers have used numerical methods as a tool to approximate the solution, e.g. finite element methods and spline interpolation.

In 1950, Hopf and Cole introduced a method to solve (2.7.1), which is known as a Hopf-Cole transformation [29]. By the Hopf-Cole transformation, a new dependent variable $w(x,t)$ is introduced such that

$$u(x,t) = -2\nu \left(\frac{\partial_x w(x,t)}{w(x,t)} \right).$$

Then, the nonlinear Burgers equation (2.7.1) is transformed to a linear heat equation

$$\partial_t w(x,t) = \nu \partial_{xx} w(x,t).$$

Since the heat equation is explicitly solvable in terms of the so-called heat kernel, then the general solution of the Burgers equation can be obtained. There are many numerical studies conducted which are relied on the Hopf-Cole transformation of the Burgers equation, e.g. [43, 44].

There are many studies have been done on the numerical methods for the Burgers equation. Numerical studies of Burgers equation by FEM can be found in [4, 25, 43]. A series of study on application of FEM and spline in approximating the Burgers equation can be found in [5, 31, 44, 60] and the references therein. Some studies on the a posteriori error estimations for the Burgers equation are studied by Patera et.al [41, 50].

Considering the importance of the Burgers equation as a mathematical model of turbulence and shock wave theory and a simplified model to study the Navier-Stokes equation, we are interested to study the error estimation of the Burgers equation. In this study, we focus on approximation of the Burgers equation without the Hopf-Cole transformation. We first implement the H1MFEM to compute the approximate solution of the Burgers equation. Secondly, we design a posteriori error estimation for the Burgers equation, using the approximate solution produced by the H1MFEM.