THE ANALYSIS OF HOMOLOGICAL FUNCTORS OF SOME TORSION FREE CRYSTALLOGRAPHIC GROUPS WITH SYMMETRIC POINT GROUP OF ORDER SIX

TAN YEE TING

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FACULTY OF SCIENCE AND MATHEMATICS
SULTAN IDRIS EDUCATION UNIVERSITY

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ABSTRACT

This study aims to analyze the homological functors of some torsion free crystallographic groups, namely Bieberbach groups, with symmetric point group of order six. The polycyclic presentations for these groups are constructed based on their matrix representations given by Crystallographic, Algorithms, and Tables package, followed by checking their consistency. The homological functors which include the nonabelian tensor square, the G-trivial subgroup of the nonabelian tensor square, the central subgroup of the nonabelian tensor square, the nonabelian exterior square, and the Schur multiplier are determined by using the computational method for polycyclic groups. The structures of the nonabelian tensor squares are explored and the generalization of the homological functors of these groups are developed up to n dimension. The findings reveal that the nonabelian tensor squares and the nonabelian exterior squares of these groups are nonabelian while the rest of the homological functors are abelian. Besides, the structures of the nonabelian tensor squares of some of these groups are found split while some are found non-split. Also, the generalizations of some homological functors, which are abelian, can be represented by the products of cyclic groups while for the homological functors which are nonabelian, their generalized presentation are constructed. In conclusion, based on the formulation of the homological functors of Bieberbach groups with symmetric point group of lowest dimension, the homological functors can be generalized up to n dimension. As the implication, this study contributes new theoretical results to the field of theoretical and computational group theory and also benefit some chemists and physicists who are interested in crystallography and spectroscopy.

ANALISIS FUNGTOR-FUNGTOR HOMOLOGI BEBERAPA KUMPULAN KRISTALOGRAFI BEBAS KILASAN DENGAN KUMPULAN TITIK SIMETRI BERPERINGKAT ENAM

ABSTRAK

Kajian ini bertujuan menganalisis fungtor-fungtor homologi bagi beberapa kumpulan kristolografi bebas kilasan, iaitu kumpulan Bieberbach, dengan kumpulan titik simetri berperingkat enam. Persembahan polikitaran bagi kumpulan tersebut dibina berdasarkan persembahan-persembahan matriks yang diperoleh daripada pakej *Crystallography, Algorithms, and Tables* diikuti dengan menyemak kekonsistenannya. Fungtor-fungtor homologi yang merangkumi kuasa dua tensor tak abelan, sub-kumpulan G-remeh bagi kuasa dua tensor tak abelan, sub-kumpulan pusat bagi kuasa dua tensor tak abelan, kuasa dua peluaran tak abelan dan pekali Schur ditentukan dengan menggunakan kaedah pengiraan bagi kumpulan-kumpulan polikitaran. Struktur-struktur bagi kuasa dua tensor tak abelan turut diterokai dan generalisasi bagi semua fungtor homologi bagi kumpulan tersebut telah dibina sehingga ke dimensi n. Dapatan kajian menunjukkan bahawa kuasa dua tensor tak abelan dan kuasa dua peluaran tak abelan bagi kumpulan tersebut adalah tidak abelan manakala fungtor-fungtor homologi yang lain adalah abelan. Selain daripada itu, struktur-struktur bagi kuasa dua tensor tak abelan bagi sesetengah kumpulan telah didapati berpisah manakala sesetengah didapati tidak berpisah. Tambahan lagi, generalisasi bagi sesetengah fungtor homologi yang abelan telah dapat diwakilkan dalam bentuk pendaraban kumpulan-kumpulan kitaran, manakala bagi fungtor homologi yang tak abelan pula, persembahan generalisasi telah berjaya dibina. Kesimpulannya, berdasarkan formulasi fungtor-fungtor homologi bagi kumpulan Bieberbach berdimensi terendah dengan kumpulan titik simetri, fungtor-fungtor homologi tersebut dapat digeneralisasi hingga ke dimensi n. Implikasinya, kajian ini menyumbang teori baru kepada bidang teori dan pengiraan teori kumpulan dan juga memberi faedah kepada ahli kimia dan ahli fizik yang berminat dalam kristalografi dan spektroskopi.

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LIST OF SYMBOLS

G^{ab}	Abelianization of G, G'/G
$Z(G)$	Center of the group G
$\nabla(G)$	Central subgroup of nonabelian tensor square of G
$[g, h]$	Commutator of g and h
h^g	Conjugate of h by g
C_n	Cyclic group of order n
G'	Derived subgroup of G
$G \times H$	Direct product of G and H
$\in$	Element of
$G \wedge G$	Exterior square of G
${F_n}^{ab}$	Free abelian group of rank n
$G \cong H$	G is isomorphic to H
$>$	Greater than
$\geq$	Greater than or equal to
$\langle X \vert R \rangle$	Group presented by generators X and relators R
$\langle x \rangle$	Group generated by the element x
$J(G)$	G-trivial subgroup of nonabelian tensor square of G
$H \leq G$	H is a subgroup of G

$H \lhd G$	H is normal in G
$B_i(j)$	i^{th} Bieberbach group with symmetric point group of order six with dimension j
g^{-1}	Inverse of element g
$\mathbb{Z}$	Integers
$\cong$	Isomorphic
gH, Hg	Left coset, and right coset of H, respectively, with coset representative g
$<$	Less than
$\leq$	Less than or equal to
$G \otimes G$	Nonabelian tensor square of G
$\notin$	Not element of
$\neq$	Not equal to
$\subset$	Proper subset
G/H	Quotient group of G by H
$M(G)$	Schur multiplier of G
$\subseteq$	Subset
$G \tilde{\otimes} G$	Symmetric square of G
S_3	Symmetry group of order six

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Torsion free crystallographic groups are called Bieberbach groups. They are the extensions of a free abelian group L of finite rank by a finite group P. Hence, there is a short exact sequence $1 \rightarrow L \xrightarrow{\sigma} G \xrightarrow{\omega} P \rightarrow 1$ such that $G/\sigma(L) \cong P$. L is called the lattice group and G is called the Bieberbach group with point group P and lattice subgroup L. The dimension of the Bieberbach group is the rank of the lattice subgroup.

Bieberbach groups have many interesting algebraic properties. The properties of the groups can be explored by computing their homological functors. The homological functors of a group G such as the nonabelian tensor square, $G \otimes G$, the G-trivial subgroup of the nonabelian tensor square, $J(G)$, the central subgroup of the nonabelian tensor square, $\nabla(G)$, the nonabelian exterior square, $G \wedge G$, the Schur multiplier, $M(G)$, the subgroup of $J(G)$, $\Delta(G)$, the symmetric square, $G \widetilde{\otimes} G$, and the G-trivial subgroup of the symmetric square, $\widetilde{J}(G)$, are originated from Algebraic K-theory. Most of the homological functors of a group G discussed above are presented in the commutative diagram with exact rows and all columns represent central extensions as given in the following theorem.

Theorem 1.1 (Bacon & Kappe, 2003)

Let G be a group. Then the rows are exact in the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J(G) & \longrightarrow & G \otimes G & \xrightarrow{\kappa} & G' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & \tilde{J}(G) & \longrightarrow & G \widetilde{\otimes} G & \xrightarrow{\kappa''} & G' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & M(G) & \longrightarrow & G \wedge G & \xrightarrow{\kappa'} & G' & \longrightarrow & 1
\end{array}
$$

where $\kappa(g \otimes h) = \kappa''(g \widetilde{\otimes} h) = \kappa'(g \wedge h) = [g, h]$.

In Theorem 1.1, G' denotes the derived subgroup of a group G. The $G \otimes G$ is generated by the symbols $g \otimes h$ where it is a specialization of the nonabelian tensor product which is defined if the two groups act on each other in a compatible way and their actions are taken to be conjugation. In addition to that, $\nabla(G)$ is generated by the elements $x \otimes x$ for $x \in G$ while $\Delta(G)$ is generated by the elements $(x \otimes y)(y \otimes x)$

for $x, y \in G$. Furthermore, the $G \wedge G$ is the quotient subgroup of the $G \otimes G$ by the $\nabla(G)$ while the $G \widetilde{\otimes} G$ is the quotient subgroup of the $G \otimes G$ by the $\Delta(G)$. In Theorem

1.1, it is clearly shown that $J(G)$ is isomorphic to the kernel of the homomorphism $\kappa : G \otimes G \rightarrow G'$, $M(G)$ is isomorphic to the kernel of the homomorphism $\kappa' : G \wedge G \rightarrow G'$, and $\tilde{J}(G)$ is isomorphic to the kernel of the homomorphism $\kappa'' : G \widetilde{\otimes} G \rightarrow G'$.

1.2 Research Background

The study of the homological functors of a group has been initiated by Brown and Loday (1987). Brown, Johnson, and Robertson (1987) has determined the homological functors of some finite groups. The other famous researchers are Bacon (1994) who focused on nilpotent groups, Ellis and Leonard (1995) who focused on Burnside groups, Beuerle and Kappe (2000) who focused on metacyclic group, and Eick and Nickel (2008) who focused on polycyclic groups.

The computation of the homological functors of Bieberbach groups with certain point group has just started in year 2008 (Rohaidah Masri, Noraini Aris, Nor Haniza Sarmin, & Morse, 2008). The homological functors of Bieberbach groups with abelian point group, particularly with cyclic point group of order two, three, and five, have been determined (Hazzirah Izzati Mat Hassim, 2014; Hazzirah Izzati Mat Hassim, Nor Haniza Sarmin, Nor Muhainiah Mohd Ali, Rohaidah Masri, & Nor'ashiqin Mohd Idrus, 2013a, 2013b, 2014; Rohaidah Masri, 2009; Rohaidah Masri et al., 2008). Moreover, the homological functors of Bieberbach groups with nonabelian point group, particularly with dihedral point group of order eight, have also been computed (Wan Nor Farhana Wan Mohd Fauzi, 2015; Wan Nor Farhana Wan Mohd Fauzi, Norashiqin

Mohd Idrus, Rohaidah Masri, & Nor Haniza Sarmin, 2013; Nor'ashiqin Mohd Idrus, 2011; Nor'ashiqin Mohd Idrus & Nor Haniza Sarmin, 2010). In this research, the homological functors of Bieberbach groups with another nonabelian point group, which is the symmetric point group of order six, are explored.

1.3 Problem Statements

Bieberbach groups are torsion free crystallographic groups. Any new findings concerning these groups will give benefits to the mathematicians, physicists and chemists. The properties of the Bieberbach groups with cyclic point group and Bieberbach groups with dihedral point group have been explored by computing their homological functors (Hazzirah Izzati Mat Hassim, 2014; Wan Nor Farhana Wan Mohd Fauzi et al., 2013; Nor'ashiqin Mohd Idrus, 2011; Rohaidah Masri, 2009). There is still no research which focuses on the homological functors of Bieberbach groups with symmetric point group (Hazzirah Izzati Mat Hassim, 2014). Thus, this research focuses on exploring the properties of Bieberbach groups with symmetric point group of order six, S_3. In this research, the homological functors of Bieberbach groups with point group S_3 such as the $\nabla(G)$, $G \otimes G$, $G \wedge G$, $J(G)$, and $M(G)$ are determined.

Rohaidah Masri (2009) constructed the generalization of the nonabelian tensor square for Bieberbach groups with cyclic point group of order two. The generalizations of the other homological functors of Bieberbach groups with cyclic point group have also been constructed (Rohaidah Masri, Hazzirah Izzati Mat Hassim, Nor Haniza

Sarmin, & Nor'ashiqin Mohd Idrus, 2014a, 2014b; Hazzirah Izzati Mat Hassim et al., 2014). Thus, the main focus in this research is to construct the generalization of the homological functors of Bieberbach groups with point group S_3. However, the generalization of the homological functors of Bieberbach groups with certain point groups constructed in the previous researches focused only on the homological functor which is abelian. So, this research aims to construct the generalization of the homological functor which is nonabelian of Bieberbach groups with symmetric point group of order six.

1.4 Research Objectives

The objectives of this research are to:

(i) formulate and explore the structure of the nonabelian tensor square of some Bieberbach groups with point group S_3,

(ii) compute and formulate some other homological functors of some Bieberbach groups with point group S_3,

(iii) develop the generalization of the formulation of some homological functors of some Bieberbach groups up to n dimension with point group S_3,

(iv) develop the algorithm in GAP in order to compute some homological functors of all Bieberbach groups up to n dimension with point group S_3,

(v) identify the limitation of the findings using GAP and using hand calculation.

1.5 Significance of the Study

The major contribution of this research will be the new theoretical results on computing the homological functors of Bieberbach groups with symmetric point group of order six. New theorems on the determination of the derived subgroup, abelianization, and various homological functors of Bieberbach groups with symmetric point group of order six are constructed in this research. Also, the generalizations of the formulation of homological functors of the groups are constructed up to n dimension. Thus, this research contributes new findings in Group Theory.

The findings in this research can be used for further research in related area. For example, Rocco and Rodrigues (2016) applied the finding of the nonabelian tensor square of nilpotent groups of class two to compute the q-tensor square of the group. Rashid, Sarmin, Erfanian, and Mohd Ali (2011) applied the findings of the homological functors of groups of order p^2q to determine the capability of the group. Besides that, the homological functors such as the Schur multiplier have been used in the computation of the classification of quasicrystals (Fisher & Rabson, 2003) and in the study of string theory (Feng, Hanany, He, & Prezas, 2001).

1.6 Scope of the Study

This research focuses on the computations of the homological functors of some Bieberbach groups with symmetric point group of order six. In identifying the independent generators and constructing the generalization of the homological functors, the groups are limited to four selected Bieberbach groups with symmetric point group of order six up to n dimension.

1.7 Thesis Organization

There are nine chapters in this thesis. Chapter 1 provides the introduction of the thesis. This chapter discusses research background, problem statements, research objectives, scope of the study, and significance of findings.

In Chapter 2, some definitions and concepts on the homological functors and Bieberbach groups are presented. Besides, some past studies regarding the Bieberbach groups and the homological functors of some groups are overviewed and discussed. This chapter also introduces the background and application of the Crystallographic Algorithms, and Tables (CARAT) and the Groups, Algorithms, and Programming (GAP). Some important definitions and preliminary results that are used throughout the thesis are also included in this chapter.

Chapter 3 presents the methodology of this research. It includes the research design and procedure, and data collection and analysis.

Chapter 4 and Chapter 5 discuss the findings of the first objective. In these chapters, the polycyclic presentations of four Bieberbach groups with point group S_3 are constructed. The derived subgroups, abelianizations, $\nabla(G)$, and nonabelian tensor squares of these four groups are also computed. Chapter 4 discusses the nonabelian tensor squares of two Bieberbach groups with point group S_3 in dimension four, given as $B_1(4)$ and $B_2(4)$. The nonabelian tensor squares of $B_1(4)$ and $B_2(4)$ are shown nonabelian and non-split. Meanwhile, Chapter 5 discusses the nonabelian tensor squares of another two Bieberbach groups with point group S_3 in dimension five, given as $B_3(5)$ and $B_4(5)$. The nonabelian tensor squares of $B_3(5)$ and $B_4(5)$ are shown nonabelian and split. In other words, their nonabelian tensors squares can be written as the direct products of their $G \wedge G$ with their $\nabla(G)$.

Chapter 6 focuses on the findings of the second objective where $G \wedge G$, $J(G)$, and $M(G)$ of the groups $B_1(4)$, $B_2(4)$, $B_3(5)$, and $B_4(5)$ are computed. The $G \wedge G$ of these groups are nonabelian while the $J(G)$ and $M(G)$ of these groups are abelian.

Chapter 7 presents the main results which gives the findings of the third objective in this thesis. The polycyclic presentations of the four Bieberbach groups with point group S_3 are constructed up to n dimension. The derived subgroups, abelianizations, and homological functors of these four groups are also generalized up to n dimension for both abelian and nonabelian cases.

Chapter 8 presents the GAP's algorithms and discusses the limitation and strength of GAP. These findings are for the last two objectives in this thesis. The verification of the formulas constructed for the group $B_1(n)$ in Chapter 7 are shown by taking example of the group of dimension 10, given as $B_1(10)$. Some findings for the abelian homological functors explored by GAP are also presented in this chapter.

Lastly, Chapter 9 presents the summary and conclusion of the research. Some suggestions for future research are also given in this chapter.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, the previous researches regarding the Bieberbach groups and the homological functors of some groups are reviewed. This chapter also includes some basic concepts and preliminary results on the homological functors of groups, especially the Bieberbach groups, that will be used throughout this research. Besides, some background on CARAT and GAP software used in assisting the computation of the homological functors of groups are also included.

2.2 Bieberbach Groups

In this section, some basic definitions and concepts on the Bieberbach groups are presented. First, Bieberbach groups are defined by the following definitions.

Definition 2.1 **Torsion-Free Group** (Robinson, 1993)

A group is said to be torsion-free if all its elements have infinite order except the identity.

Definition 2.2 **Bieberbach Group** (Hiller, 1986)

A Bieberbach group G is a torsion free group given by a short exact sequence $1 \to L \to G \to P \to 1$ where L is a free abelian normal subgroup of G of finite rank, called lattice of G, P is finite group called point group of G, and G/L is isomorphic to P. The point group G acts on L by conjugation in G.

A crystallographic group is a discrete subgroup G of the set of isometries of Euclidean space E^n, where the quotient space E^n/G is compact (Hiller, 1986). Some properties of the crystallographic group are given in the following theorem and proposition.

Theorem 2.1 (Hiller, 1986)

An abstract group G is isomorphic to an n-dimensional crystallographic group if and only if G contains a finite index, normal, free abelian subgroup of rank n.

Proposition 2.1 (Hiller, 1986)

A crystallographic group G is a Bieberbach group if and only if it is torsion free.

Hiss and Sczepanski (1991) explored the torsion free crystallographic groups. Opgenorth, Plesken, and Schulz (1998) created the tables for a computer package called Crystallographic, Algorithms, and Tables (CARAT), which can handle crystallographic groups up to dimension six. Plesken and Schulz (2000) computed the number of crystallographic groups in dimension five and six. Cid and Schulz (2001) computed and classified the five and six dimensional torsion free crystallographic groups. They showed that there are 1060 Bieberbach groups in dimension five, of which 101 have trivial center and there are 38746 Bieberbach groups in dimension six, of which 5004

have trivial center.

The study on the torsion free generalized crystallographic groups with the indecomposable holonomy group which is isomorphic to either a cyclic group of order p^s, for any integer s, or a direct product of two cyclic groups of order p has been done in the past (Bovdi, Gudivok, & Rudko, 2002, 2004). Bovdi et al. (2004) showed that there are finitely many non-isomorphic indecomposable torsion free generalized crystallographic group with holonomy groups isomorphic to the alternating group of degree four.

Rohaidah Masri (2009) provided a method to create a family of Bieberbach groups from a given Bieberbach group B that have the same point group as in the

following lemma.