FORMULATION OF THE NONABELIAN TENSOR SQUARES OF SOME BIEBERBACH GROUPS WITH POINT GROUP $C_2 \times C_2$: ABELIAN AND NONABELIAN CASES

NOR FADZILAH BINTI ABDUL LADI

05-4506832 pustaka.upsi.edu.my Perpustakaan Tuanku Bainun Kampus Sultan Abdul Jalil Shah PustakaTBainun ptbupsi

THESIS SUBMITTED IN FULFILMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE (MATHEMATICS)
(MASTER BY RESEARCH)

FACULTY OF SCIENCE AND MATHEMATICS
UNIVERSITI PENDIDIKAN SULTAN IDRIS

2018

05-4506832 pustaka.upsi.edu.my Perpustakaan Tuanku Bainun Kampus Sultan Abdul Jalil Shah PustakaTBainun ptbupsi

ABSTRACT

The main objective of this research is to compute the nonabelian tensor squares of some Bieberbach groups with elementary abelian 2-groups point groups, $C_2 \times C_2$ for both abelian and nonabelian cases. This is qualitative research where the computational method for polycyclic groups is used to determine the nonabelian tensor squares of four Bieberbach groups with point group $C_2 \times C_2$. The findings show that the structures of the nonabelian tensor squares of some of these groups for both abelian and nonabelian cases are found split while some are found non-split. The formulas of the abelian cases of the nonabelian tensor squares of the groups lead to the construction of the formulations of the nonabelian tensor squares of arbitrary dimension. Furthermore, for the nonabelian cases, the presentations of the nonabelian tensor squares of the other two groups are given and it was shown that the nonabelian tensor squares can be written as a direct product with the nonabelian exterior square as one of the factor. As a conclusion, the formulas of the nonabelian tensor squares of four Bieberbach groups with point group $C_2 \times C_2$ are developed based on its group presentations of lowest dimension and the formulas of the abelian cases can be generalized up to dimension n. As the implication, this research contributes new theoretical results in the field of theoretical and computational group theory on computing the nonabelian tensor squares of Bieberbach groups with elementary abelian 2-group point group. The results in this research will also benefit to other group theorist who are interested of the computation of the computation of the homological functors.

FORMULASI TENSOR KUASA DUA TAK ABELAN BAGI KUMPULAN BIEBERBACH DENGAN KUMPULAN TITIK $C_2 \times C_2$ BAGI KES ABELAN DAN TAK ABELAN

ABSTRAK

Objektif utama kumpulan ini adalah untuk mengira tensor kuasa dua tak abelan bagi kumpulan Bieberbach dengan kumpulan abelan asas 2-kumpulan, $C_2 \times C_2$ sebagai kumpulan titik untuk kedua-dua kes abelan dan tak abelan. Penyelidikan ini merupakan penyelidikan kaulitatif yang menggunakan kaedah pengiraan bagi kumpulan polikitaran digunakan untuk menentukan tensor kuasa dua tak abelan bagi empat kumpulan Bieberbach dengan kumpulan titik $C_2 \times C_2$. Dapatan penyelidikan menunjukkan bahawa struktur tensor kuasa dua tak abelan sebahagian kumpulan bagi kes abelan dan tak abelan adalah berpisah dan sebahagian kumpulan tidak berpisah. Rumus tensor kuasa dua tak abelan kumpulan tersebut bagi kes abelan membawa kepada pembinaan formula bagi tensor kuasa dua tak abelan untuk sebarang dimensi. Selanjutnya, bagi kes tak abelan, persembahan tensor kuasa dua tak abelan bagi dua kumpulan lain diberikan dan ia menunjukkan bahawa kuasa dua tensor tak abelan boleh ditulis sebagai hasil darab langsung dengan kuasa dua perluaran sebagai salah satu faktor. Sebagai kesimpulannya, rumus bagi tensor kuasa dua tak abelan bagi empat kumpulan Bieberbach dengan kumpulan titik $C_2 \times C_2$ dihasilkan berdasarkan persembahan kumpulan tersebut bagi dimensi yang terendah dan rumus bagi kes abelan dilanjutkan sehingga dimensi n. Implikasinya, penyelidikan ini menyumbang kepada dapatan teori baru dalam bidang teori kumpulan dan teori kumpulan pengiraan dalam menentukan tensor kuasa dua tak abelan bagi kumpulan Bieberbach dengan kumpulan titik kumpulan abelan asas 2-kumpulan. Dapatan daripada penyelidikan ini memberi kelebihan kepada ahli teori kumpulan lain yang berminat dalam menentukan fungtor homologi.

TABLE OF CONTENTS

		Page
DECLARATION		ii
ACKNOWLEDGEMENT		iii
ABSTRACT		iv
ABSTRAK		v
TABLE OF CONTENTS		vi
LIST OF TABLES		ix
LIST OF FIGURES		x
LIST OF SYMBOLS		xi
CHAPTER 1	**INTRODUCTION**	1
	1.1 Introduction	1
	1.2 Research Background	3
	1.3 Problem Statements	4
	1.4 Research Objectives	5
	1.5 Scope of the Study	6
	1.6 Significance of Findings	7
	1.7 Conclusion	7
CHAPTER 2	**LITERATURE REVIEW**	8
	2.1 Introduction	8

2.2 The Nonabelian Tensor Squares of Groups 9

2.3 Some Basic Definitions, Concepts and Notation Used 14

2.4 The Nonabelian Tensor Squares of Bieberbach Group with Certain Point Group 29

2.5 CARAT and GAP 31

2.6 Conclusion 34

CHAPTER 3 METHODOLOGY 35

3.1 Introduction 35

3.2 Research Design and Procedure 36

3.3 Operational Framework 38

3.4 Conclusion 39

CHAPTER 4 THE NONABELIAN TENSOR SQUARES OF BIEBERBACH GROUPS WITH POINT GROUP $C_2 \times C_2$: ABELIAN CASES 40

5.1 Introduction 40

5.2 Computing the Nonabelian Tensor Squares of the Group $S_1(3)$ 41

5.3 Computing the Nonabelian Tensor Squares of the Group $S_2(3)$ 64

5.4 Conclusion 82

CHAPTER 5 THE NONABELIAN TENSOR SQUARES OF BIEBERBACH GROUPS WITH POINT GROUP $C_2 \times C_2$: NONABELIAN CASES 83

5.1 Introduction 83

5.2 Computing the Nonabelian Tensor Squares of the Group $S_3(4)$ 84

5.3 Computing the Nonabelian Tensor Squares of the Group $S_4(4)$ 107

5.4 Conclusion 126

CHAPTER 6 GENERALIZATION OF THE NONABELIAN TENSOR SQUARES OF BIEBERBACH GROUP $S_1(3)$ AND $S_2(3)$ 127

6.1 Introduction 127

6.2 The Nonabelian Tensor Squares of Group $S_1(n)$ 128

6.3 The Nonabelian Tensor Squares of Group $S_1(n)$ 136

6.4 Conclusion 144

CHAPTER 7 CONCLUSION 145

7.1 Summary of the Research 145

7.2 Implication of the Research 147

7.3 Suggestion for Future Research 147

REFERENCES 149

APPENDICES 152

LIST OF TABLE

Table No. **Page**

3.1 Research Procedure 36

LIST OF FIGURE

Figure No.		Page
2.1	The commutative diagram	10

LIST OF SYMBOLS

G^{ab}	Abelianization of G, G/G'
$Z(G)$	Center of the group G
$[g,h]$	Commutator of g and h
${}^{g}h$	Conugate of h by g
C_n	Cyclic group of order n
C_0	Cyclic group of ininite order
G'	Derived subgroup of G
$G \times H$	Direct product of G and H
$\in$	Element of
$G \wedge G$	Exterior square of G
F_n^{ab}	Free abelian group of rank n
$G \cong H$	G is isomorphic to H
$\langle X \mid R \rangle$	Group presented by generators X and relators R
$\langle x \rangle$	Group presented by the element x
$H \leq G$	H is a subgroup of G
$H \lhd G$	H is normal in G
$S_i(j)$	i^{th} Bieberbach group with cyclic point group of order 2 with dimension j
gH, Hg	Left coset and right coset of H, respectively with coset representative g
$G \otimes G$	Nonabelian tensor square of G

$\nabla(G)$	Central subgroup of the nonabelian tensor square
$\notin$	Not element of
$\neq$	Not equal to
$\|G\|,\|x\|$	Order of the group G, the order of the element x
$\subset$	Proper subset
G/H	Quotient group of G by H
$\subseteq$	Subset
$H_2(G)$	Second homology group
Γ	Whitehead quadratic functor

CHAPTER 1

INTRODUCTION

1.1 Introduction

Bieberbach groups are torsion free crystallographic groups which has no elements of finite order. These groups are extensions of a finite point group P and a free abelian group L of finite rank. Then, there is a short exact sequence

$$1 \rightarrow L \xrightarrow{\varphi} G \xrightarrow{\phi} P \rightarrow 1$$

such that $G/\varphi L \cong P$. Here, L is called a lattice group and P is point group, also known as a holonomy group. Hiller (1986) has conducted a research on crystallography and cohomology of groups. Bieberbach group is also known as a crystallographic group which has no elements of finite order. In 2001, Cid and Schulz (2001) were focused on the computation and the classification of five and six dimensional torsion free crystallographic groups. The basis of an algorithm that decides torsion-freeness of a crystallographic group as well as the triviality of its centre was described clearly. For a group G the nonabelian tensor square $G \otimes G$ is generated by the symbols $g \otimes h$ for all $g, h \in G$, subject to relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g} h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h} h')$$

for all $g, g', h, h' \in G$ where ${}^{g}g' = gg'g^{-1}$ denotes the left conjugate of g' by g.

The nonabelian tensor square of the group, $G \otimes G$, which was introduced by Brown and Loday (1987), is a specialization of the more general nonabelian tensor product. This group construction has its roots in algebraic K-theory and topology which is an extended ideas of Whitehead. The nonabelian tensor square appears independently in Dennis (1976) works in K-theory and is based on the ideas of Miller (1952).

The computation of the nonabelian tensor square of group G leads to a simple or standard form for expressing $G \otimes G$. The definition of the nonabelian tensor square gives no insight as to the group it describes or its structures. The nonabelian tensor squares are one of the important elements on computing homological functors of groups. The method of computing the nonabelian tensor square in computational group theory has been started by Brown, Johnson and Robertson (1987) and has been used to investigate problems related to the nonabelian tensor squares of groups. Many papers on computing the nonabelian tensor squares for various groups and classes of groups have emerged since the publication by Brown et al. (1987) seminal work. These include 2-generator nilpotent of class 2 (Bacon & Kappe, 2003; Nor Haniza, 2002; Kappe, Nor Haniza & Visscher, 1999), metacyclic groups (Beuerle & Kappe, 2000) and free nilpotent groups (Blyth, Moravec & Morse, 2008).

1.2 Research Background

In this research, the Bieberbach groups G with cyclic point of order $C_2 \times C_2$ up to dimension six are considered. The study of the computation for nonabelian tensor squares of Bieberbach groups had been started by Rohaidah (2009) and followed by Nor'ashiqin (2011), Hazzirah Izzati (2014), Wan Nor Farhana (2015) and recently Tan Yee Ting (2016). Rohaidah (2009) investigated the nonabelian tensor squares of Bieberbach groups with cyclic point group of order two, C_2 for both abelian and nonabelian cases. The results showed that the formula obtained for the abelian nonabelian tensor square could be extended to calculate the nonabelian tensor squares of Bieberbach groups with point group C_2 of arbitrary dimension, n. While for the nonabelian cases, the nonabelian tensor squares of some Bieberbach groups with cyclic point of order two and elementary abelian 2-group are showed to be nilpotent of class two and could be written as a direct product of the central subgroup of the nonabelian tensor square of the group and the exterior square of the group.

In 2011, Nor'ashiqin computed the nonabelian tensor square of centerless Bieberbach group of dimension four with the dihedral point group of order eight. Hazzirah Izzati (2014) continued Rohaidah's work by conducting a research on the computation of the homological functors of Bieberbach groups with cyclic point groups of order two, three and five. She found that for all Bieberbach groups with cyclic point groups of order three and five, the nonabelian tensor squares and exterior squares are always nonabelian. Wan Farhana (2015) conducted a research on computing the homological functors of some Bieberbach groups with dihedral point group. Recently, the computation of the nonabelian tensor squares of a Bieberbach group with symmetric point group up to order six had been determined (Tan Yee Ting et al., 2016a; Tan Yee Ting et al., 2016b).

In this research, the works of Rohaidah (2009) are continued by conducting a research on nonabelian tensor squares of Bieberbach groups with point group $C_2 \times C_2$ up to dimension 6 for both abelian and nonabelian cases.

1.3 Problem Statements

Bieberbach groups are torsion free crystallographic groups. The findings concerning these groups will give benefits to mathematicians, physicists and chemists. Rohaidah (2009) found that all Bieberbach groups with cyclic point group of order two, C_2 were shown to be metabelian and for nonabelian cases their nonabelian tensor squares are nilpotent of class at most 2. From the findings, she found that there are eleven families of Bieberbach groups with cyclic point, C_2. However, there only two families have nonabelian tensor squares which are abelian. The point group of $C_2 \times C_2$ is chosen to determine the structure of the nonabelian tensor square from the extension group of point group C_2.

Based on the results in Rohaidah (2009), it is found that the nonabelian tensor squares are abelian and nonabelian. Meanwhile, all the nonabelian tensor squares in Nor'ashiqin (2011), Hazzirah Izzati (2014), Wan Nor Farhana (2015) and Tan Yee Ting et al. (2016a) are found to be nonabelian. The various differences of results on nonabelian tensor squares give a motivation in this research to identify the structure of the nonabelian tensor squares for Bieberbach group with point group $C_2 \times C_2$. Also, it is rare to find the nonabelian tensor square which is abelian for groups of infinite order. The results in Rohaidah (2009) are used as a guide in order to compute the nonabelian tensor squares of some Bieberbach groups with point group $C_2 \times C_2$. Group, Algorithms and Programming (GAP, 2015) software is used to generate some patterns of examples while constructing the generalization of the formula of the nonabelian tensor squares of the groups for arbitrary dimension, n for abelian cases.

1.4 Research Objectives

The objectives of this research are :

(i) To determine four matrix groups of some Bieberbach groups with point group $C_2 \times$

C_2.

(ii) To determine the consistent polycyclic presentations of four Bieberbach groups in objective (i).

(iii) To compute the nonabelian tensor squares of Bieberbach groups as given in objective (ii).

(iv) To determine whether the nonabelian tensor squares of groups found in objective (iii) are abelian or nonabelian.

(v) To generalize the formula of the nonabelian tensor squares for abelian cases that have been found in objective (iii) for arbitrary dimension, n.

1.5 Scope of the Study

In this research, some Bieberbach groups with point group $C_2 \times C_2$ up to dimension six are considered. The groups are chosed based on their abelian property of the nonabelian tensor square.

1.6 Significance of the Study

The major contribution of this research are the new theoretical results on computing the nonabelian tensor squares of Bieberbach groups with point group $C_2 \times C_2$. The findings give the generalization of the formula in computing the nonabelian tensor squares of these groups of an arbitrary dimension. Thus, this research provides new results in the field of group theory. The results obtained can be used for further research in related areas such as the computation of the homological functors of these groups and the generalization of the homological functors up to n dimension.

1.7 Conclusion

This chapter gives the introduction related to the nonabelian tensor squares of some Bieberbach groups. This chapter discussed the researches related to the computation of the nonabelian tensor squares of some groups with certain point groups which are used as a guide in this research. This research focuses on the nonabelian tensor squares of some Bieberbach groups with point group of $C_2 \times C_2$ up to dimension six. This chapter contains the research background, the problem statement of this research, research objectives, research questions, scopes of study and significance of the study.

CHAPTER 2

LITERATURE REVIEW

05-4506832 pustaka.upsi.edu.my Perpustakaan Tuanku Bainun Kampus Sultan Abdul Jalil Shah PustakaTBainun ptbupsi

2.1 Introduction

This chapter discusses past researches on the nonabelian tensor squares of some various groups and the nonabelian tensor squares of some Bieberbach groups with certain point groups. The method of computing the nonabelian tensor square of a group is briefly described and some background on Crystallographic, Algorithms and Tables (`CARAT`) and Groups, Algorithms and Programming (`GAP`) software package used in this research are reviewed. Some related lemmas, propositions, corollaries and theorems are also given in this chapter.

05-4506832 pustaka.upsi.edu.my Perpustakaan Tuanku Bainun Kampus Sultan Abdul Jalil Shah PustakaTBainun ptbupsi

2.2 The Nonabelian Tensor Squares of Groups

In this section, the development of the computation of the nonabelian tensor squares for various groups and the methods used are given. Some previous researches related to the nonabelian tensor square of groups are discussed.

Homological functors were originated from homotopy theory, the study of homotopy groups which are used in algebraic topology to classify topological spaces. The definitions of the homological functors considered in this research are given as in Figure **??**. The nonabelian tensor squares is one of the homological functors. The nonabelian tensor product $G \otimes H$ of groups G and H introduced by Brown and Loday (1987) is generated by the symbols $g \otimes h$, for all $g \in G, h \in H$, subject to relations,

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

for all for all $g, g', h, h' \in G$ where ${}^{g}g' = gg'g^{-1}$ and G acts on itself by conjugation. The nonabelian tensor squares $G \otimes G$ of a group G is a special case of the nonabelian tensor product $G \otimes H$, where $G \otimes H$ is defined if both two groups act on each other in a compatible way. Blyth, Morse and Redden (2004) computed the nonabelian tensor square for the free 2-Engel group of rank $n > 3$. The nonabelian tensor square of one of the finite groups homomorphic image, namely the Burnside group of rank n and exponent 3 were also computed using the method developed in Ellis and Leonard (1995).

The main goal of computing the nonabelian tensor square is to present the

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \Gamma(G^{ab}) & \xrightarrow{\psi} & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & & & \kappa\downarrow & & \kappa'\downarrow & & \\
 & & & & G' & \xrightarrow{=} & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Figure 2.1. The Commutative Diagram

presentation of $G \otimes G$ in terms of its generators. The structure of $G \otimes G$ in terms of its central extensions has been investigated by Brown and Loday (1987). Their findings showed that the group $\nabla(G) = \{g \otimes g | g \in G\}$ is a central subgroup of $G \otimes G$ while the factor group $(G \otimes G)/\nabla(G)$ is the nonabelian exterior square of G is denoted by $G \wedge G$. Let $J_2(G)$ denotes the kernel of the mapping $G \otimes G \to G' : g \otimes h \mapsto [g, h]$ which G' is the derived subgroup of G and let Whitehead's quadratic functor denotes by $\Gamma(G/G')$ where (G/G') is the abelianization of G, G^{ab} and the kernel of $G \wedge G \to G' : g \otimes h \mapsto [g, h]$ is isomorphic to the Schur multiplier, $H_2(G)$ (Whitehead, 1950; Brown & Loday, 1987). Most of the homological functors of a group G discussed above are described in the following commutative diagram (Brown et al., 1987) as given in Figure 2.1.

For the infinite groups, the method used in the computation of the nonabelian tensor square is known as the crossed pairing method. The standard technique in the computation of the nonabelian tensor square for infinite group is to determine a mapping $\Phi : G \times G \to L$, where G and L are groups. Here, Φ is called a crossed pairing if Φ

satisfies the following two conditions,

$$\Phi(gg', h) = \Phi({}^{g}g', {}^{g}g)\Phi(g, h) \text{ and } \Phi(g, hh') = \Phi(g, h)\Phi({}^{h}g, {}^{h}h')$$

for all g, g', h, h' in G. The crossed pairing method determines a unique homomorphism $\Phi^* : G \otimes G \rightarrow L$. Bacon (1994) used cross pairing method to compute the nonabelian tensor square for the free nilpotent groups of class two of finite rank while Beuerle and Kappe (2000) used the method to compute the nonabelian tensor squares of the infinite metacyclic groups. Also, this method had been used in computing the nonabelian tensor square for abelian cases such as two-generator two-groups of class two, (Kappe et al., 1999; Nor Haniza, 2002) and the free 2-Engel groups of finite rank, (Bacon & Kappe, 2003; Blyth et al., 2004). The used of crossed pairing methods is more difficult if the nonabelian tensor square of the group is not abelian. The method used by Ellis and Leonard (1995) was extended by Blyth and Morse (2009) for infinite groups and polycyclic groups to overcome the limitations of crossed pairing method. By using the polycyclic method, Blyth and Morse provide a general commutator calculus in group G which helps in the computations of $G \otimes G$.

Rocco (1991) has conducted a research on a new construction related to the nonabelian tensor square of a group. He investigated the properties of the group $\nu(G)$ as defined in the following definition.

Definition 2.1 $\nu(G)$ (Rocco, 1991)

Let G be a group with presentation $\langle \mathcal{G} \mid \mathcal{R} \rangle$ and let G^{φ} be an isomorphism copy of G via the mapping $\varphi : g \to g^{\varphi}$ for all$g \in G$. The group $\nu(G)$ is defined to be

$$\nu(G) = \langle \mathcal{G}, \mathcal{G}^{\varphi} \mid \mathcal{R}, \mathcal{R}^{\varphi}, {}^{x}[g, h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x^{\varphi}}[g, h^{\varphi}], \forall x, g, h \in G \rangle.$$

Rocco (1991) has developed the commutator calculus associated with the subgroup $[G, G^{\varphi}]$ of $\nu(G)$. The subgroup $[G, G^{\varphi}]$ is a fully invariant subgroup of $\nu(G)$. Based on this findings, Ellis and Leonard (1995) showed that the commutator subgroup of $\nu(G)$ is isomorphic to the nonabelian tensor square of the group G as given in the following theorem.

Theorem 2.1 (Ellis & Leonard, 1995)

Let G be group. The map $\phi : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$ defined by $\phi(g \otimes h) = [g, h^{\varphi}]$ for all g, h in G is an isomorphism.

The above theorem tells us that all the tensor computations can be done by using the commutator computation within the subgroup $[G, G^{\varphi}]$ in $\nu(G)$ since there is an isomorphism from $G \otimes G$ into $[G, G^{\varphi}]$.

Moreover, Ellis and Leonard (1995) used the computational method to construct a relatively small finite presentation of $\nu(G)$, computed a concrete presentation for $\nu(G)$ and applied standard computational group theory methods to compute the nonabelian tensor square of the Burnside group $B(2,4)$ of order 2^{12}.

The method of computing the nonabelian tensor squares of polycyclic groups has been developed by Blyth and Morse (2009). They proved that if G is polycyclic, then $G \otimes G$ is polycyclic and is so $\nu(G)$, as stated in the following proposition.

Proposition 2.1 (Blyth & Morse, 2009)

If G is polycyclic, then $\nu(G)$ is polycyclic.

Since $G \otimes G$ is isomorphic to $[G, G^{\varphi}]$ which is normal in $\nu(G)$ by Theorem 2.1, and Blyth and Morse has proved that $\nu(G)$ is polycyclic if G is polycyclic as shown in Proposition 2.1, hence the nonabelian tensor square can be computed by using the polycyclic method.

Blyth and Morse (2009) computed the nonabelian tensor square by hand calculations aided by the following Proposition 2.2 which gives a complete description of a set of generators of $[G, G^{\varphi}]$.

Proposition 2.2 (Blyth & Morse, 2009)

Let G be polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1, ..., \mathfrak{g}_k$. Then $[G, G^{\varphi}]$, a subgroup of $\nu(G)$, is generated by $[\mathfrak{g}_i, \mathfrak{g}_i]$, $[\mathfrak{g}_i^{\epsilon}, (\mathfrak{g}_j^{\varphi})^{\delta}]$ and $[\mathfrak{g}_i^{\epsilon}, (\mathfrak{g}_j^{\varphi})^{\delta}][\mathfrak{g}_j^{\delta}, (\mathfrak{g}_i^{\varphi})^{\epsilon}]$, that is

$$[G, G^{\varphi}] = \langle [\mathfrak{g}_i, \mathfrak{g}_i^{\varphi}], [\mathfrak{g}_i^{\epsilon}, (\mathfrak{g}_j^{\varphi})^{\delta}], [\mathfrak{g}_i^{\epsilon}, (\mathfrak{g}_j^{\varphi})^{\delta}][\mathfrak{g}_j^{\delta}, (\mathfrak{g}_i^{\varphi})^{\epsilon}] \rangle$$

for $1 \leq i < j \leq k$, where

$$\epsilon = \begin{cases} 1 & \text{if } |\mathfrak{g}_i| < \infty; \\ \pm 1 & \text{if } |\mathfrak{g}_i| = \infty \end{cases}$$

and

$$\delta = \begin{cases} 1 & \text{if } |\mathfrak{g}_j| < \infty; \\ \pm 1 & \text{if } |\mathfrak{g}_j| = \infty. \end{cases}$$

Hence, based on the methods given from the previous researches, it is shown that the polycyclic method is the effective method on computing the nonabelian tensor square compared to others methods. Hence, this method is used in this research.

2.3 Some Basic Definitions, Concepts and Notations Used

In this section, some definitions, lemmas and theorems related to this research are given. Firstly, the definition of Bieberbach group and some basic concept related to this research are given as in the following:

Definition 2.2 Bieberbach Group (Hiller, 1986)

A bieberbach group G is a torsion free group given by a short exact sequence

$$1 \rightarrow L \xrightarrow{\varphi} G \xrightarrow{\phi} P \rightarrow 1$$