THE SYSTEM OF EQUATIONS FOR MIXED BVP OF PDE WITH CONSTANT COEFFICIENT

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ABSTRACT

The aim of this research is to produce the system of equations for three different mixed Boundary Value Problems (BVPs). The potential problem which involves the Laplace's equation on a square shape domain was considered, where the boundary is divided into four sets of linear boundary elements. The Boundary Element Method (BEM) was used to approximate the solutions for BVP. The mixed BVPs were reduced to Boundary Integral Equation (BIEs) by using direct method which were related with Green's second identity representation formula. Then, linear interpolation was used on the discretized elements. The results showed that, there are three system of equations which were obtained. For some cases of mixed BVPs which involves discontinuous fluxes problems yields underdetermined systems. Out of the three problems that being considered, one of three BVPs leads to the underdetermined system of equations. Therefore, the transformation for the underdetermined system to the standard form is necessary for the numerical purposes. The gradient approach method which is widely applies to the Dirichlet problem was considered. This gradient approach method is extended to the underdetermined system of equations obtained from the mixed BVP which subsequently transformed to the standard system. In conclusion, the mixed BVP that involve discontinuous fluxes problem will yield to the underdetermined system of equations that prohibits in solving the system numerically. However, by the gradient approach method, the underdetermined system can be transformed to the standard form and can be solved the system numerically. The study implicates that the procedure used in this studies can be extended to higher dimensional mixed BVPs which involved the discontinuous fluxes problems.

SISTEM PERSAMAAN UNTUK MASALAH NILAI SEMPADAN CAMPURAN TERHADAP PERSAMAAN PEMBEZAAN SEPARA DENGAN PEKALI MALAR

ABSTRAK

Tujuan penyelidikan ini untuk menghasilkan sistem persamaan bagi tiga Masalah Nilai Sempadan (MNS) campuran yang berbeza. Masalah potensi yang melibatkan persamaan Laplace pada domain bentuk persegi dipertimbangkan di mana sempadan dibahagikan kepada empat set elemen linear. Kaedah Unsur Sempadan (KUS) digunakan untuk penyelesaian dalam MNS. Formula identiti kedua Green digunakan untuk menjadikan MNS campuran kepada Persamaan Sempadan Kamiran (PSK). Kemudian, interpolasi linear pula digunakan pada elemen yang telah dibahagikan. Dapatan kajian menunjukkan bahawa tiga sistem persamaan linear boleh diperoleh. Untuk beberapa kes MNS bercampur yang melibatkan masalah sudut, ia menghasilkan sistem yang tidak jelas. Daripada tiga masalah yang dipertimbangkan, salah satu masalah menimbulkan sistem yang tidak jelas. Oleh itu, transformasi untuk sistem yang tidak jelas kepada sistem persamaan piawai adalah diperlukan untuk tujuan berangka. Satu kaedah telah dipertimbangkan iaitu kaedah pendekatan kecerunan yang mana kaedah ini secara meluas digunakan untuk masalah Dirichlet. Dalam kajian ini, kaedah pendekatan kecerunan telah digunakan kepada sistem persamaan MNS bercampur yang tidak jelas yang mana seterusnya sistem standard boleh diperoleh. Kesimpulannya, MNS bercampur yang melibatkan masalah fluks yang tidak berterusan akan menghasilkan sistem persamaan yang tidak jelas yang menghalang sistem untuk diselesaikan secara berangka. Walau bagaimanapun, dengan kaedah pendekatan kecerunan, sistem yang tidak jelas dapat berubah menjadi bentuk standard dan membolehkan sistem diselesaikan secara berangka. Kajian ini membuktikan bahawa prosedur yang digunakan dalam kajian ini boleh diperluaskan kepada MNS campuran yang berdimensi lebih tinggi bagi sistem yang melibatkan masalah fluks yang tidak berterusan.

CONTENTS

		PAGE
DECLARATION		ii
AKNOWLEDGMENT		iii
ABSTRACT		iv
ABSTRAK		v
CONTENTS		vi
LIST OF TABLES		ix
LIST OF FIGURES		x
LIST OF ABBREVIATIONS		xi
LIST OF APPENDIX		xii
CHAPTER 1	**INTRODUCTION**	
1.1	Introduction	1
1.2	Background of the Study	3
1.3	Problem Statement	4
1.4	Objectives	6
1.5	Research Questions	7
1.6	Research Rationale	8
1.7	Outline of Thesis	8

CHAPTER 2 LITERATURE RIVIEW

2.1 Introduction 10

2.2 Overview of Research 11

2.3 Conclusion 16

CHAPTER 3 METHODOLOGY

3.1 Introduction 17

3.2 Boundary Integral Equation 18

3.3 The System of Equations for Mixed BVP with Constant Coefficient 24

3.3.1 The descretization of the BIE 25

3.3.2 Assembly the system of equations 29

3.4 Conclusion 30

CHAPTER 4 RESULTS AND FINDINGS

4.1 Introduction 32

4.2 A system of equations related to mixed BVP with one prescribed value for Dirichlet BC and three prescribed values for Neumann BCs. 33

4.3 A system of equations related to mixed BVP with two prescribed values for Dirichlet BCs and two prescribed values for Neumann BCs. 39

4.4 A system of equations related to the mixed BVP with one prescribed value for Neumann BC and three prescribed values for Dirichlet BCs 44

4.5 Conclusion 55

CHAPTER 5 CONCLUSION

5.1 Summary of the Research 57

5.2 Implication of the Research 60

5.3 Suggestion for Future Research 60

REFERENCES 61

LIST OF TABLES

Table No.		Page
4.1	Integrals of known and unknown values in each elements and nodes for Problem 1	35
4.2	Integrals of known and unknown values in each elements and nodes for Problem 2	40
4.3	The classification of known and unknown values on each elements and nodes for Problem 3	47

LIST OF FIGURES

No. Figures **Page**

3.1 The domain Ω, normal ν, boundary $\partial\Omega_N$ and $\partial\Omega_D$ 18

3.2 The chart that explained the process in solving the mixed BVP by using BEM 30

4.1 The considered domain with mixed BC with one prescribed value for Dirichlet BC and three prescribed values for Neumann BCs 33

4.2 Example of mixed boundary condition with two Neumann prescribed values and two Dirichlet prescribed values 38

4.3 The considered domain with mixed BC with one Neumann prescribe value and three Dirichlet prescribe values 45

4.4 Fluxes at a corner node as described in Paris and Canas (1997) 49

LIST OF ABBREVIATIONS

BEM	Boundary Element Method
BVP	Boundary Value Problem
FEM	Finite Element Method
PDE	Partial Differential Equation
BC	Boundary Condition

APPENDIX LIST

A The System of Equations for Mixed BVP with Three Dirichlet Boundary Conditions and One Neumann Boundary Condition

B The System of Equation for Mixed BVP with One Dirichlet Boundary Condition and Three Neumann Boundary Conditions

CHAPTER 1

INTRODUCTION

1.1 Introduction

A Boundary value problem (BVP) is a Partial Differential Equation (PDE) with prescribe boundary conditions imposed at different points. An Initial value problem (IVP) is a PDE with prescribed initial condition. Many physical problems are associated by second order PDE's such as fluid mechanics, heat transfer, rigid body dynamics and elasticity. BVPs of PDE can be solved by two ways which are by analytical method and numerical method. There have many analytic method in solving BVPs such as separation of variables, Laplace and Faurier transforms, variation of parameters and integral transform. However, analytical method can only solve the PDE with simple boundary conditions. In other words, analytic method cannot handle all PDEs. Therefore, numerical method is a good option to approximate the solution for BVP. The examples of numerical methods used to solve BVPs are Finite Difference Method, Finite Element Method, Finite Volume Method, volume discretization and Boundary Element Method. The most common method use is Finite Element Method (FEM). But, there is another method that can be considered, which is a Boundary Element Method (BEM). The advantage of BEM is the data preparation for BEM are simple. In some cases of BVPs such as those involving infinite domains or 3-dimensional complicated domains, the BEM is more preferable method in solving BVP as compared to FEM. One of the advantage is the BEM reduces the problem dimensionality by one which the discretization of two dimensional (2D) problem only involves the boundary of domain and the discretization for three dimensional (3D) problem only involves the domain of geometry. Unlike FEM, that requires the discretization of the entire domain in 2D problem. As this advantage, it will reduce the computational time in solving the problem. As compared to FEM, the BEM is less time consuming process, but this method also involves some disadvantages. The first disadvantage is BEM has difficulty since the integrals involve singular integrands. Furthermore, the BEM requires the good knowledge of a suitable fundamental solution for a particular equation.

As is well known, a PDE can be reduced to a Boundary Integral Equations (BIE). The BVPs are very widely used in the real world problems and most of the engineers and scientists use BIE in solving the BVPs (Ali & Rajakumar, 2004). The examples of the application of BIE are potential flow calculations, crack problems in elasticity, electrostatic and elastostatic calculations. There are two ways in the process of reduction of the BVPs for PDE with constant coefficients to BIEs. The first way is a direct method which based on the second Green's identity representation formula whereas the second way is indirect method which is based on the single or double layer potentials.

1.2 Background of the Research

This research is concentrated on forming the systems of equations of mixed BVP for PDE with constant coefficients with one prescribed value for Dirichlet BC and three prescribed values for Neumann BCs, two prescribed values for Dirichlet BCs and two prescribed values for Neumann BCs and one prescribed value for Neumann BC and three prescribed values for Dirichlet BCs. We focus for the mixed BVPs of PDE which is combination of Dirichlet boundary condition and Neumann boundary condition. We apply BEM to approximate the solutions of PDE. In this research, we consider a two dimensional potential problem which satisfies Laplace equation. For reformulation of BVPs, we use direct method which involve Greens identity representation formula in the process of reformulation from mixed BVPs to BIE.

In this thesis, we consider a square shape domain as our test domain with mixed boundary conditions. The BEM can be applied to curve domain such as a circle domain. However, we are not interested to the curve domain because there is no corner nodes on the smooth boundary. We have three considered problems, the discretization of the BIE with one prescribed value for Dirichlet BC and three prescribed values for Neumann BCs, the discretization of the BIE with two prescribed values for Dirichlet BCs and two prescribed values for Neumann BCs and the discretization of the BIE with one prescribed value for Neumann BC and three prescribed values for Dirichlet BCs. The boundary of the domain is divided into small linear segments. Then, linear interpolation method is used on the discretized elements. In considered mixed BVP with one prescribed value for Neumann BC and three prescribed values for Dirichlet BCs, the system of equations yield from the discretized BIE is underdetermined system. Therefore, in our research, we also discuss an example of corner node problem on mixed BVP. The corner node problem is a situation when there are two fluxes unknowns remain associated to a node due to discontinuous outward normal. For solving the corner node problem in mixed BVP, we study a gradient approaches method which explained by Paris and Canas (1997) to produce a standard system of equations.

1.3 Problem Statement

In Brebbia (1977), showed the discussion of the BEM for potential problem that involved corner node problem. However, the system of equations for the problems are not presented in his work and the method used in solving the corner node problem is by making the assumption that there are two nodes closeness to the corner node. In Menin and Rolnik (2013) have discussed the system of equations obtained from mixed BVP. However, the obtained system of equations is in standard form.

We can find the discussions on corner node problem in example Gaul, Kogl, and Wagner (2003) and Sedaghatjoo, Dehghan and Hosseinzadeh (2013). The method used by Gaul et al. (2003) to solve the corner nodes problems due to the discontinuous fluxes are by using discontinuous elements. There are two ways in discontinuous elements method, the first way is by using constant elements and another way is by using double nodes. Besides that, the method used by Sedaghatjoo et al. (2013) to solve the discontinuous fluxes is finite difference approach method.

In Paris and Canas (1997) discussed a method to solve the corner node problems that is gradient approach method. However, they only apply the proposed method for Dirichlet problem only.

In this work, we consider three different mixed BVPs. The mixed BVP that involve the discontinuous fluxes problem is yields the underdetermined system of equations. The numerical purpose cannot be done if the obtained system of equations for mixed BVP is not in standard form. Therefore, in this work, we extend the gradient approach method to overcome the discontinuous fluxes for mixed BVP and automatically, the underdetermined system of equations can be transformed to the standard system.

1.4 Objectives

The objectives of this research are:

1. To produce a system of equations yield from the discretized BIE for mixed BVP with constant coefficient involved of corner node problem on a square shape domain with one prescribed value for Dirichlet BC and three prescribed values for Neumann BCs.
2. To produce a system of equations yield from the discretized BIE for mixed BVP with constant coefficient involved of corner node problem on a square shape domain with two prescribed values for Dirichlet BCs and two prescribed values for Neumann BCs.
3. To produce a system of equations yield from the discretized BIE for mixed BVP with constant coefficient involved of corner node problem on a square shape domain with one prescribed value for Neumann BC and three prescribed values for Dirichlet BCs.
4. To transform the system of equations obtained in objective 3 to the standard form by using gradient approach method in Paris and Canas (1997) on the discontinuous flux problem for mixed BVP.

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1.5 Research Questions

The research questions of this research are:

1. What is the system of equations produced from the discretized BIE for mixed BVP with constant coefficient involved of corner node problem on a square shape domain with one prescribed value for Dirichlet BC and three prescribed values for Neumann BCs.
2. What is the system of equations produced from the discretized BIE for mixed BVP with constant coefficient involved of corner node problem on a square shape domain with two prescribed values for Dirichlet BCs and two prescribed values for Neumann BCs.
3. What is the system of equations produced from the discretized BIE for mixed BVP with constant coefficient involved of corner node problem on a square shape domain with one prescribed value for Neumann BC and three prescribed values for Dirichlet BCs.
4. Is it possible to use gradient approach method in Paris and Canas (1997) on the transform the system of equations obtained in objective 3 to the standard form and what is the method use to handle the discontinuous flux problem for mixed BVP.

1.6 Research Rationale

The BVPs have wider applications in physics phenomenon. The BEM is a preferable method to solve the BVPs since it is less time consuming for approximating the solutions. Therefore, it will benefit the scientist to use the BEM in solving a BVP with PDE.

1.7 Outline of Thesis

This thesis is presented of five chapters. The five chapters of this thesis are the introduction, literature review, methodology, discussion, and conclusion. We start with an introduction of boundary element method in chapter 1. In this chapter, we presented the background of our topic, objectives of this study, research rationale, our scope of study and outline of this thesis.

In chapter 2, the overview of this topic is mentioned. BVPs are very widely used in the physics phenomenon. Therefore, the research on the BVP had started on centuries ago. The discussion on the BIE, BEM, mixed BVPs and corner node problem are also described in this section.

In the next chapter, the concepts of BEM which is used to solve mixed BVPs for PDE with constant coefficient is discussed. The mixed BVP is reduced to a boundary integral equation by using Greens second identity representation formula. Then, a discretization of the boundary, $\partial\Omega$ of our considered domain is carried out by using linear elements. The linear interpolation is used on the discretized elements so that numerical solution can be provided to solve the integrals. After that, an assembling process is discussed which the system of equations obtained is written in matrix form. Here, we consider a square shape domain related to three different prescribed mixed boundary conditions position. These three problems will be discussed in this chapter. Furthermore, the gradient approaches method is introduced in order to solving the mixed BVP which involve corner node problems. The details of the gradient approaches method is also discussed in this chapter.

In chapter 4 and 5, the results from the systems of equations obtained in three selected problems are presented. The three considered problems are related to the mixed BVPs. Each problem will come out with a system of equations. However, the assembly process for the system of equations related to the mixed BVP is not be as straight forward as Dirichlet BVP and Neumann BVP. Among these problems, the mixed BVP in problem 3 produce an underdetermined matrix due to the corner node problems. Therefore, we apply the gradient approaches method from Paris and Canas (1997) which used only by them for Dirichlet BVP to overcome the corner node problems and automatically will come out with a standard system of equations.

In the last chapter, some conclusions and summarization of this thesis are given. Furthermore, the suggestion for further study is also written in this chapter.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The BVPs are very widely used in the physics phenomenon. The BVP can be reduced to the BIE by using direct method and indirect method. The study of the BIEs had been started long time ago before the BEM come out as a numerical tools to approximate the solution of BIE. In this chapter, the background and overview of the research are explained.

2.2 Overview of Research

In 1828, George Green was the first that used the term potential function in electricity and also produced the Green's function and Green's Theorem in his book entitled "An Essay on the Application of Mathematical Analysis to the theories of Electricity and Magnetism" (Green, 1828). Green's identities are basically used in the formulation process of BEM.

The discussion on the fundamental solutions can be found in Costabel and Stephan (1985). The weighting residual technique is applied in the formulation of BIE. Hence, the weighting function used in the process is called as fundamental solution of equation in order to reduce the BVP to BIE. The fundamental solutions are very well known and the knowledge of the fundamentals of BEM is important.

The discussion on the BIE was explored by Brebbia and Domingnez (1977), and Aliabadi (2002). In these books, the formulation of BIE from BVP on transmission problem and potential problem in 2 and higher dimension for the Helmholtz or Laplace equation was discussed. The direct method which based on Green's formula is used to derive the system of BIEs.

The formulation of BIE from the PDE through the indirect method was discussed in Atkinson and Chandler (1990), Maz'ya and Soloviev (2003). In the case of the Dirichlet and Neumann problems for the Laplace equation, the reduction process of the

BVPs to BIE are represented in the form of simple and double layer potentials. Furthermore, the study of existence and uniqueness of solutions to the problems was explored by Bremer and Rokhlin.

In 1994, the discussion of the BEM for two dimensional potential problems was explored by Hall (1994). The author was considered the boundary integral equation for the different boundary conditions such as potential boundary condition, flux boundary condition and mixed boundary condition. Since all physical phenomenan occurring in nature can be described by differential equations and boundary conditions nowadays so there must very useful to have the programming of BEM to solve the PDEs. This will make the engineer and scientist more easily to solve problem.

The discussion about FEM and BEM were showed by Hunter and Pullan (2003) and Ali and Rajakumar (2005). They presented the numerical method to solve the BIE which they discretized the boundary of domain into small segments such as constant, linear and curve elements for 2D. The interpolation function was applied on each discretized elements. This led to a system of equations and the integral value for each segment was assembled into a global matrix which can be solved numerically.

The BEM with programming was written by Paris and Canas (1997), Ang (2007), Beer, Smith, and Duenser (2008). They explained about the BEM method on potential problem and elasticity problem with two and three dimensional. In a book of Beer et al. (2008) entitled “The Boundary Element Method with Programming”, the integrals of each segment was solved by using Gauss Quadrature method. However, the integrals that involve singularity, it was solved by using Gauss Laguerre method. The programming code was developed in Fortran software for every process in BEM had showed to solve the PDEs with constant coefficient. However, for the PDE with mixed boundary condition needs some modification in the integral equation.

In 1977, Brebbia published his work (Brebbia, 1977) about the discussion of the BEM for potential problems. In this paper, the author was used the weighted residual technique to form the boundary element formulation. The numerical method was applied to the simple examples by using a square domain with three different solution, for example, approximate the solution of equation by using constant elements, linear elements and linear elements with two nodes at corners. This paper also involved the corner node problem due to the discontinuous normal values at a corner node but the system matrix was not shown in this paper. Therefore, to handle this problem, the author was assumed there are two points closeness to the corner node but belonging to different sides of discontinuous linear elements.

The discussion of the BEM fundamentals and applications was discussed by Paris and Canas (1997). In this book, the authors were discussed the theory of BEM and the numerical implementation of this theory was introduced in order to solve the actual and complicated problems. In this book, Paris and Canas (1997) was mentioned the disadvantage of corner nodes in Dirichlet problem. Hence, they introduced the gradient approaching method to solve the disadvantage in Dirichlet problem.

In addition, Gaul, Kogl, and Wagner (2003) wrote a book entitled "Boundary Element Methods for Engineers and Scientists" that discussed the general procedure and application of the BEM to potential theory and elastic theory. The authors showed the comparison between the boundary element and finite element formulations. The discontinuous fluxes in mixed BVP also discussed in this book. In order to handle the discontinuous fluxes, the authors considered two methods to solve the problem that are by using discontinuous elements such as constant elements and double nodes method.

The discussion about the spectral properties for the BIE obtained from the direct method was explored by Mohamed (2013). Thus, the results indicate that the related spectral properties obtained for the direct method related to the Dirichlet BVP and Neumann BVP also lies within the unit circle. However, there is no spectral properties for the PDE with mixed BVP in this jurnal Mohamed (2013). Boundary Domain Integral Equation (BDIE) for PDEs with variable coefficient also discussed by author.

The application of BEM on the free surface flow was discussed in Makau, Gathia, and Manyanga (2016) paper. The authors solve the BIE of free surface flow by numerical method. The numerical example is applied on a circle domain with different number of nodal points. Furthermore, the authors show the comparison of numerical solution and exact solution results. The numerical integration and matrix computations were calculated by using MATLAB software.