CHROMATICITY OF K_4-HOMEOMORPHS WITH GIRTH 9 AND 6-BRIDGE GRAPHS

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The chromaticity of graphs is the term used referring to the question of chromatic equivalence and chromatic uniqueness of graphs. Since the arousal of the interest on the chromatically equivalent and chromatically unique graphs, various concepts and results under the said areas of research have been discovered and many families of such graphs have been obtained. The purpose of this thesis is to contribute new results on the chromaticity of graphs, specifically, K_4-homeomorphs with girth 9 and 6-bridge graphs. A K_4-homeomorph is a graph derived from a complete graph, K_4. Such a homeomorph is denoted by $K_4(a, b, c, d, e, f)$ where the six edges of K_4 are replaced by the six paths of length a, b, c, d, e and f. Let $\mathbb{N}$ and θ_k be a set of natural numbers and a multigraph with two vertices and k edges, respectively. For any $a_1, a_2, \ldots, a_k \in \mathbb{N}$, the graph $\theta(a_1, a_2, \ldots, a_k)$ is a sub-

division of θ_k where the edges of θ_k are replaced by paths of length $a_1, a_2, \ldots, a_k$, respectively. The subdivision of θ_k is called a multi-bridge graph or a k-bridge graph. The results in this thesis cover two main parts. The first part involves the chromaticity of K_4-homeomorphs with girth 9 and the second part discusses the chromaticity of 6-bridge graphs. We first study the chromaticity of a type of K_4-homeomorphs with girth 9, that is, the graph $K_4(2,3,4,d,e,f)$. We then investigate the chromaticity of another type of K_4-homeomorphs with girth 9, that is, the graph $K_4(1,4,4,d,e,f)$. Then, we obtain the complete solution for the chromaticity of all types of K_4-homeomorphs with girth 9. For the latter part, we first investigate the chromaticity of 6-bridge graph $\theta(3,3,3,b,b,c)$ where $3 \leq b \leq c$. We next study the chromaticity of 6-bridge graph $\theta(a,a,a,b,b,c)$ where $2 \leq a \leq b \leq c$. We continue to determine the chromaticity of 6-bridge graph $\theta(a,a,b,b,b,c)$ where $2 \leq a \leq b \leq c$. Next, we investigate two more types of 6-bridge graphs of the form $\theta(3,3,b,b,c,c)$ where $3 \leq b \leq c$ and $\theta(3,3,3,b,c,d)$ where $3 \leq b \leq c \leq d$. Many new results on the chromaticity of K_4-homeomorphs with girth 9 and 6-bridge graphs are obtained. We end this thesis by including some open problems for further investigation.

Abstrak tesis yang dikemukakan kepada Senat Universiti Malaysia Terengganu sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KEKROMATIKAN GRAF K_4-HOMEOMORFIK DENGAN KITAR TERPENDEK 9 DAN GRAF 6-JAMBATAN

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Kekromatikan bagi sesuatu graf adalah suatu terma yang merujuk kepada persoalan tentang graf-graf yang setara secara kromatik dan unik secara kromatik. Sejak minat terhadap graf-graf yang setara secara kromatik dan unik secara kromatik semakin berkembang, pelbagai konsep dan penemuan berkenaan bidang ini diperolehi dan beberapa keluarga bagi graf-graf ini telah ditemui. Tujuan tesis ini adalah untuk menyumbang kepada keputusan-keputusan baru bagi kekromatikan sesuatu graf, khususnya graf K_4 dengan kitar terpendek 9 dan graf 6-jambatan. K_4-homeomorfik adalah suatu graf yang diterbitkan daripada graf lengkap, K_4. Homeomorfik ini diwakilkan dengan $K_4(a, b, c, d, e, f)$ yang keenam-enam sisi K_4 digantikan dengan enam lintasan yang panjang masing-masing adalah a, b, c, d, e dan f. Andaikan $\mathbb{N}$ dan θ_k masing-masing adalah suatu set nombor asli dan

graf pelbagai dengan dua bucu dan k sisi. Untuk sebarang $a_1, a_2, \ldots, a_k \in \mathbb{N}$,

$\theta(a_1, a_2, \ldots, a_k)$ adalah subbahagian bagi θ_k yang sisi-sisi θ_k digantikan dengan lintasan-lintasan yang panjangnya $a_1, a_2, \ldots, a_k$. Subbahagian bagi θ_k ini dipanggil graf pelbagai-jambatan atau graf k-jambatan. Keputusan dari tesis ini meliputi dua bahagian utama. Bahagian pertama melibatkan kekromatikan graf K_4-homeomorfik dengan kitar terpendek 9 dan bahagian kedua membincangkan kekromatikan graf 6-jambatan. Pertama sekali, kami membincangkan kekromatikan satu jenis graf K_4-homeomorfik dengan kitar terpendek 9, iaitu graf $K_4(2,3,4,d,e,f)$. Kemudian, kami menyiasat kekromatikan satu lagi jenis graf K_4-homeomorfik dengan kitar terpendek 9, iaitu graf $K_4(1,4,4,d,e,f)$. Seterusnya, kami mendapatkan penyelesaian lengkap bagi kekromatikan semua graf K_4-homeomorfik dengan kitar terpendek 9. Di dalam bahagian seterusnya, kami menyiasat kekromatikan graf 6-jambatan $\theta(3,3,3,b,b,c)$ dengan $3 \leq b \leq c$. Seterusnya, kami menyelidik kekromatikan graf 6-jambatan $\theta(a,a,a,b,b,c)$ dengan $2 \leq a \leq b \leq c$. Kami teruskan dengan menentukan kekromatikan graf 6-jambatan $\theta(a,a,b,b,b,c)$ dengan $2 \leq a \leq b \leq c$. Kemudian, kami menyiasat dua jenis graf 6-jambatan berbentuk $\theta(3,3,b,b,c,c)$ dengan $3 \leq b \leq c$ dan $\theta(3,3,3,b,c,d)$ dengan $3 \leq b \leq c \leq d$. Banyak keputusan baru untuk kekromatikan graf K_4-homeomorfik dengan kitar terpendek 9 dan graf 6-jambatan telah diperolehi. Kami akhiri tesis ini dengan memberikan beberapa masalah terbuka untuk kajian akan datang.

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LIST OF ABBREVIATIONS AND NOTATIONS

i.e.	that is
h.r.p.	highest remaining power
l.r.p.	lowest remaining power
$\mathbb{N}$	a set of natural numbers
$d_G(v)$	degree of vertex v in G
$e(G)$	size of G
$v(G)$	order of G
$E(G)$	set of edges in G
$V(G)$	set of vertices in G
$N_G(v)$	neighbourhood of vertex v in G
$P(G,\lambda)$ or $P(G)$	chromatic polynomial of graph G
$Q(G,s)$ or $Q(G)$	essential polynomial of graph G
$H \subseteq G$	H is a subgraph of G
K_4	complete graph with four vertices
(n,m)-graph	graph with order n and size m
$\Delta(G)$	maximum degree of G
$\delta(G)$	minimum degree of G
θ_k	multigraph with two vertices and k edges
$\square$	end of proof

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Graph theory aroused for the first time in 1735 when Leonhard Euler succesfully solved the problem of The Seven Bridges of Konigsberg. The Four-Colour Problem is the problem in determining whether four colours are sufficient in order to colour any maps so that neighbouring countries have different colours.

This problem was first conjectured by Francis Guthrie in 1852 and was brought forwarded to De Morgan before been presented by Arthur Cayley to the London Mathematical Society in 1878. Since then several studies have been done by many researchers in order to solve this problem including Birkhoff [4], who introduced a new function called a chromatic polynomial of M, denoted by $P(M, \lambda)$. This function is defined as the number of proper λ-coloring of a map M using at most λ distinct colours. The study on the chromatic polynomial has progressed tremendously ([5], [55], [47], [52]) and several researchers investigate the root of chromatic polynomials ([6], [3]).

Even though Birkhoff's findings could not successfully solved the Four-Colour Problem, however, it managed to stimulate the development of the knowledge on chromatic polynomial and leads to the growth of the knowledge on graph theory.

By investigating the chromatic polynomial, Read [47] came out with a problem in determining the necessary and sufficient condition for two graphs to have the same chromatic polynomial. Later on, in 1978 a definition on the chromatically unique graph has been defined by Chao and Whitehead Jr. [16] where a graph is said to be chromatically unique if there are no other graphs have the same chromatic polynomial as it is. Many studies on the chromatic uniqueness of graphs have been conducted and various results have been obtained ([35], [36], [21]). The term chromaticity problem refers to the problem involving chromatic equivalence and chromatic uniqueness of a graph.

A graph is said to be a complete graph if for every two distinct vertices in a graph, these vertices are connected by an edge. Complete graph with n vertices is denoted by K_n. A K_4-homeomorph is a subdivision of a K_4. It is denoted by $K_4(a,b,c,d,e,f)$ when six edges of K_4 are replaced by six paths of length a, b, c, d, e and f, respectively. Figure 1.1 shows the diagram of $K_4(a,b,c,d,e,f)$.

Multigraph is a graph which does not permit loops but multiple edges are allowed. Let θ_k be a multigraph with two vertices and k edges, and $\mathbb{N}$ be the set of natural numbers. For any $a_1, a_2, \ldots, a_k \in \mathbb{N}$, the graph $\theta(a_1, a_2, \ldots, a_k)$ is a subdivision of θ_k where the edges of θ_k are replaced by paths of length $a_1, a_2, \ldots, a_k$, respectively. The subdivision of θ_k is called a multi-bridge graph or a k-bridge graph. Figure 1.2 shows the diagram of k-bridge graph.

Motivated by the studies conducted by several researchers regarding the chromaticity of these two graphs, we investigated the chromaticity of K_4-homeomorphs

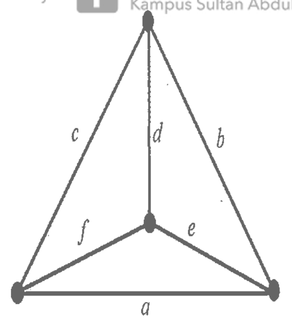


Figure 1.1: $K_4(a, b, c, d, e, f)$

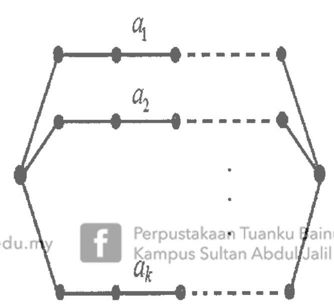


Figure 1.2: Multi-bridge graph

with girth 9 and some 6-bridge graphs.

1.2 Fundamental of Graph Theory

We will present the basic terminologies normally used in graph theory. Further explanation may refer to West [53] and Harary [25]. A graph G is an ordered pair $(V(G), E(G))$ consisting a non-empty vertex set, denoted by $V(G)$ and edge set, denoted by $E(G)$, where each edge consists of two (possibly identical) vertices called its endpoint. Each vertex is indicated by a point and each edge is indicated by a line joining the vertices. The *order* of G, denoted by $v(G)$ is the number of vertices in G while the *size* of G, denoted by $e(G)$ is the number of edges in G. Here, for $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$, the order of G is n and the size of G is m, and G is being called as a (n, m)-graph. If $u, v \in V(G)$ and there is an edge $e = uv$ such that $e \in E(G)$, then e is said to join u and v. The vertices u and v are called the ends of e, and u and v are *adjacent*. In addition, u and v are *incident* with e and vice versa, and u is a neighbour of v and vice versa. The *neighbourhood* of v, denoted by $N_G(v)$ or $N(v)$ is defined as $\{x \in V(G) : x \text{ and } v \text{ are adjacent}\}$.

A *loop* is an edge whose endpoints are identical. Parallel edges or *multiple edges* are edges that have the same pair of endpoints. A simple graph is a graph having no loops or multiple edges. A finite graph is a graph where both its vertex set, $V(G)$ and edge set, $E(G)$ are finite.

Let v be a vertex in G, then the *degree* of v denoted by $d_G(v)$ or $d(v)$ is the number of edges of G incident with v where each loop is counting as two edges. The maximum degree is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. A graph G is a regular graph if $\Delta(G) = \delta(G)$. A vertex with degree 0 is called an isolated vertex and a vertex with degree 1 is called an end-vertex.

A graph H is a *subgraph* of G, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning subgraph* is a subgraph containing all the vertices of G.

A *walk* of length n in G is an alternating sequence of vertices and edges in the form $v_0, e_1, v_1, e_2, ..., e_n, v_n$ such that $e_i = v_{i-1}v_i$ for all $1 \leq i \leq n$, and may also be denoted by $v_0v_1v_2...v_n$, and being called as a walk joining v_0 and v_n or a $v_0 - v_n$ walk. A closed walk is a walk where $v_0 = v_n$. We may say that a closed walk is a walk which it has the same endpoints.

A *trail* is a walk with no repeated edge. A closed trail is a trail where its endpoints are identical. A *path* is a trail with no repeated vertex. If the path has the same endpoints, then it is called a *cycle*.

A graph G is a *connected graph* if for each pair of vertices such that $u, v \in V(G)$ is joined by a path. Otherwise, it is disconnected. A *component* of G is a maximal connected subgraph in G. A connected graph is *Eulerian* if there is a closed trail which includes every edge of G. A connected graph is *Hamiltonian* if there is a cycle which includes every vertex of G.

A graph is an *acyclic* graph if it has no cycle. A *tree* is a connected acyclic graph or a connected graph without any cycle. A *forest* is an acyclic graph which its components are trees. Let G be a graph with at least one cycle. The *circumference* of G is the length of the longest cycle in G and the *girth* of G is the length of the shortest cycle in G.

Let G and H be two non-empty graphs. An edge-gluing of G and H is a graph obtained from G and H by identifying one edge of G with one edge of H.

1.3 Objectives of the Study

The aim of this study is to determine the necessary and sufficient conditions for the chromaticity of graph. This study is focusing on two types of graphs, that are K_4-homeomorphs with girth 9 and 6-bridge graphs. Hence, we have the following objectives to be achieved.

i. Determine the chromaticity of K_4-homeomorphs with girth 9.

ii. Investigate the chromaticity of certain types of 6-bridge graph.

1.4 Organisation of Thesis

In Chapter 1, we introduce the background of the study and give the definition of the terminologies used throughout the study. We also state the objectives to be accomplished at the end of this study and provide the organisation of the thesis. In Chapter 2, we present some fundamental results of chromatic polynomial and provide the literature studies on the chromaticity of K_4-homeomorphs and k-bridge graphs.

In Chapter 3, we discuss the chromatic equivalence pairs of a type of K_4-homeomorphs with girth 9, that is $K_4(2,3,4,d,e,f)$ and these results will lead to the chromatic uniqueness of $K_4(2,3,4,d,e,f)$. In Chapter 4, we investigate the chromatic equivalence pair and chromatic uniqueness of $K_4(1,4,4,d,e,f)$.

In Chapter 5, we determine the chromaticity of another four types of K_4-homeomorphs with girth 9, that are $K_4(1,3,5,d,e,f)$, $K_4(1,2,6,d,e,f)$, $K_4(1,2,c,3,e,3)$ and $K_4(1,3,c,2,e,3)$. In this chapter, we present the complete solution of chromatic uniqueness of K_4-homeomorphs with girth 9.

In Chapter 6, we investigate the chromaticity of a type of 6-bridge graph $\theta(3,3,3,b,b,c)$ where $3 \leq b \leq c$. We generalize this result into any 6-bridge graph $\theta(a,a,a,b,b,c)$ where $2 \leq a \leq b \leq c$ in Chapter 7.

In Chapter 8, we investigate the chromaticity of another type of 6-bridge graphs $\theta(a,a,b,b,b,c)$ where $2 \leq a \leq b \leq c$. In Chapter 9, we determine the chromaticity of two types of 6-bridge graphs, namely $\theta(3,3,b,b,c,c)$ where $3 \leq b \leq c$ and $\theta(3,3,3,b,c,d)$ where $3 \leq b \leq c \leq d$.

We end this thesis by summarizing the results obtained in Chapter 10. We shall provide some open problems to end this chapter.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we shall discuss the fundamental results on the chromatic polynomial that leads to the emergence of a special theorem that is widely used in determining the chromatic polynomial of certain graphs. As a consequence, the notions of chromatic equivalence and chromatic uniqueness shall be introduced. The preliminary studies and the literature on the chromaticity of K_4-

homeomorphs and k-bridge graphs shall be presented in the last two sections of this chapter.

2.2 The Fundamental Results on Chromatic Polynomial

Definition 2.1 *Let G be a graph. A proper colouring is a colouring such that two adjacent vertices are assigned to two distinct colours, and also known as a mapping*

$$f : \{v_1, v_2, ..., v_n\} \rightarrow \{1, 2, ..., \lambda\}$$

such that $f(v_i) \neq f(v_j)$ whenever $v_iv_j \in E(G)$ where $\{v_1, v_2, ..., v_n\} = V(G)$. Two proper λ-colourings f and g are different if $f(v_i) \neq g(v_i)$ for some vertex v_i

in G. G is said to be λ-colourable if it admits a λ-colouring.

Definition 2.2 *A chromatic number, denoted by $\chi(G)$ is the minimum number λ such that G is λ-colourable. A graph G is λ-chromatic if $\chi(G) = \lambda$. Then, let the chromatic polynomial of G, denoted by $P(G, \lambda)$ or simply $P(G)$ be the number of different proper λ-colouring of G, i.e. the number of ways that G is coloured such that the adjacent vertices are having different colours.*

Recall that the polynomial $P(M, \lambda)$ was first introduced by Birkhoff in 1912.

Theorem 2.1 *(Birkhoff [4]) The number of ways of colouring a given map, M in λ colours is*

$$P(M, \lambda) = \sum_{i=1}^{n} \lambda^i \sum_{k=0}^{n-1} (-1)^k (i, k),$$

where $i =$ *number of regions of a submap,*

$n =$ *number of regions of a map,*

$k =$ *number of joinings of the regions,*

$\lambda =$ *number of colours needed to colour the map,*

$(i, k) =$ *number of steps needed to break down a map of n regions into submap of i regions.*

However, as Birkhoff tried to solve the Four-Colour Problem on maps, Whitney tried to interpret Birkhoff's results into an arbitrary graph.

Theorem 2.2 *(Whitney [55]) Let G be a graph with n vertices and m edges. Then*

$$P(G, \lambda) = \sum_{p=1}^{n} \left(\sum_{r=0}^{m} (-1)^r N(p, r) \right) \lambda^p$$

where $N(p, r)$ is the number of spanning subgraphs of G consisting p components *and r edges.*

Let G be a graph with n vertices and m edges and $\beta : E(G) \to \{1, 2, ..., m\}$ be a bijection. For any cycle C in G, let $e \in G$ such that $\beta(e) > \beta(x)$ for any x in $E(C)\backslash\{e\}$, the path $C - e$ is the *broken cycle* in G with respect to β.

Theorem 2.3 (Whitney's Broken-Cycle Theorem) *(Whitney [55]) Let G be an (n, m)-graph, and let $\beta : E(G) \to \{1, 2, ..., m\}$ be any bijection. Then*

$$P(G, \lambda) = \sum_{i=1}^{n} (-1)^{n-i} h_i(G) \lambda^i$$

where $h_i(G)$ is the number of spanning subgraphs of G that have exactly $n - i$ edges and contain no broken cycles with respect to β.

Theorem 2.4 (Whitney's Reduction Theorem) *(Whitney [56]) Let $e \in E(G)$. Then*

$$P(G, \lambda) = P(G - e, \lambda) - P(G \cdot e, \lambda)$$

where $G - e$ is a graph obtained from G by deleting e and $G \cdot e$ is a graph obtained from G by contracting e.

Here, $P(G, \lambda)$ can be expressed in the form of the chromatic polynomial of empty graphs. The Whitney's Reduction Theorem can also be written as follow.

Let $v_i, v_j \in V(G)$ such that v_i and v_j be two non-adjacent vertices in a graph G and $v_iv_j \notin E(G)$. Then

$$P(G, \lambda) = P(G + v_iv_j, \lambda) + P(G \cdot v_iv_j, \lambda)$$

for $G + v_iv_j$ is a graph obtained from G by adding the edge v_iv_j and $G \cdot v_iv_j$ is a graph obtained from G by the coalescence of v_i and v_j.

Now, $P(G, \lambda)$ can be expressed in the form of the chromatic polynomial of complete graph which is in factorial form.

Several researchers developed the chromatic polynomial of some graphs. Suppose G_1 and G_2 be graphs each containing a complete subgraph K_r where $r \geq 1$. Zykov [62] introduced the chromatic polynomial of graph G where G is obtained from the union of G_1 and G_2 by identifying the two subgraphs K_r in an arbitrary way. Then, G is called a K_r-gluing of G_1 and G_2. A K_1-gluing is called a *vertex-gluing* and a K_2-gluing is called an *edge-gluing* of G_1 and G_2, respectively.

Lemma 2.1 *(Zykov [62]) Let G be a K_r-gluing of graphs G_1 and G_2. Then*

$$P(G) = \frac{P(G_1)P(G_2)}{P(K_r)} = \frac{P(G_1)P(G_2)}{\lambda(\lambda-1)\cdots(\lambda-r+1)}.$$

Read [47] as well found the chromatic polynomial of G when G is a disconnected graph consists of several components.

Lemma 2.2 *(Read [47]) If a graph G has connected components $G_1, G_2, ..., G_k$, then*

$$P(G) = P(G_1)P(G_2)\cdots P(G_k).$$

The following are the properties of chromatic polynomial, $P(G)$ developed by Read and Farrell.

Theorem 2.5 *(Read [47]) Let G be a graph of order n and size m. Then $P(G, \lambda)$ is a polynomial in λ such that*

i. deg $(P(G, \lambda))$ is n;

ii. all the coefficients are integers;

iii. *the constant term is zero;*

iv. *the leading term is* λ^n*;*

v. *the terms alternate in sign;*

vi. *the absolute value of the coefficient of* λ^{n-1} *is* m*;*

vii. *either* $P(G,\lambda) = \lambda^n$ *or the sum of the coefficients in* $P(G,\lambda)$ *is zero.*

Theorem 2.6 *(Farrell [23]) Let* G *be a graph of order* n *and size* m*. Then the coefficient of*

i. λ^n *is 1;*

ii. λ^{n-1} *is* $-m$*;*

iii. λ^{n-2} *is* $\binom{m}{2} - t_1(G)$*;*

iv. λ^{n-3} *is* $-\binom{m}{3} + (m-2)t_1(G) + t_2(G) - 2t_3(G)$*;*

where $t_1(G) =$ *the number of triangles (complete graph with three vertices,* K_3*) in* G*,*

$t_2(G) =$ *the number of cycles with order four without chords in* G*,*

$t_3(G) =$ *the number of complete graph with four vertices* (K_4) *in* G*.*

2.3 Chromatic Equivalence and Chromatic Uniqueness of Graphs

Definition 2.3 *Let* G *and* H *be two graphs. Then,* G *and* H *are said to be chromatically equivalent (or simply* χ*-equivalent), denoted by* $G \sim H$ *if both graphs have the same chromatic polynomials, that is* $P(G,\lambda) = P(H,\lambda)$*.*

Definition 2.4 *Two graphs* G *and* H *are said to be isomorphic, denoted by*

$G \cong H$ *if there exist a bijective function* $\varphi : V(G) \rightarrow V(H)$ *such that* $uv \in E(G)$

if and only if $\varphi(u)\varphi(v) \in E(H)$. *Such a mapping is called an isomorphism of* G *onto* H.

Definition 2.5 *A graph* G *is said to be chromatically unique (or simply* χ*-unique) if for any graph* H, $H \cong G$ *whenever* $H \sim G$. *The relation* $\sim$ *is an equivalence relation on the class of graphs. The chromatic equivalence class determined by* G *under* $\sim$ *is denoted by* $[G]$, *indeed* $[G]$ *is the set of all the graphs having the same chromatic polynomial,* $P(G, \lambda)$. *Apparently,* G *is* χ*-unique if and only if* $[G] = \{G\}$.

Corollary 2.1 *Let* G *be a graph. Then the chromatic number of* G, $\chi(G)$ *is the smallest positive integer* λ *such that* $P(G) > 0$.

The following theorem gives the necessary conditions for two graphs to be χ-equivalent.

Theorem 2.7 *(Koh & Teo [35]) Let* G *and* H *be two* χ*-equivalent graphs. Then,*

i. $v(G) = v(H)$;

ii. $e(G) = e(H)$;

iii. $\chi(G) = \chi(H)$;

iv. $t_1(G) = t_1(H)$;

v. $t_2(G) - 2t_3(G) = t_2(H) - 2t_3(H)$;

vi. G *is connected if and only if* H *is connected;*

vii. G *is 2-connected if and only if* H *is 2-connected;*

viii. $g(G) = g(H)$;

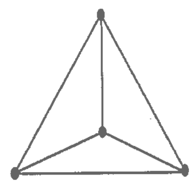
ix. G *and* H *have the same number of cycles with length equal to their girth;*

where $g(G)$ = girth of G,

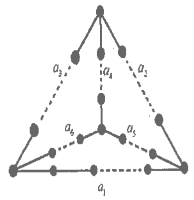
$g(H)$ = girth of H.

2.4 Chromaticity of K_4-Homeomorphs

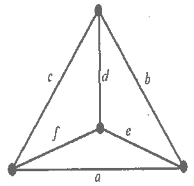
Harary [25] defined two graphs are *homeomorphic* if both can be obtained from the same graph by a sequence subdivision of lines, that is divisions of edges into several segments. A K_4-homeomorph is a subdivision of K_4 by replacing the edges in K_4 by paths of length a, b, c, d, e, f, respectively, and be denoted by $K_4(a, b, c, d, e, f)$. Figure 2.1(a) shows the diagram of K_4 graph, while Figure 2.1(b) and Figure 2.1(c) are two ways in representing K_4-homeomorphs. However, we shall use Figure 2.1(c) (as in Figure 1.1) to refer to the general form of K_4-homeomorphs.



(a)



(b)



(c)

Figure 2.1: (a) K_4, (b) $K_4(a_1, a_2, a_3, a_4, a_5, a_6)$, (c) $K_4(a, b, c, d, e, f)$

Definition 2.6 *A path $v_0v_1...v_k$ is called a chain if $d_G(v_i) = 2$ for each $i = 1, 2, ..., k-1$, and is a maximal chain if, in addition, $d_G(v_0) \geq 3$ and $d_G(v_k) \geq 3$.*

The length of a chain P is denoted by $l(P)$. The six paths of K_4-homeomorphs are its maximal chains.

According to Chao and Zhao [17], the chromaticity of K_4-homeomorphs was first studied by Kahn in 1980 in his doctoral dissertation entitled "Chromatic equivalence and chromatic uniqueness". Chao and Zhao managed to obtain the chromatic polynomials for connected $(n, n+2)$-graphs and came out with Lemma 2.3 as follow.

Lemma 2.3 *(Chao & Zhao [17]) Any graph that is χ-equivalent to K_4-homeomorphs must itself be a K_4-homeomorphs.*

The chromatic polynomial of K_4-homeomorphs can be determined using the Whitney's Reduction Theorem (Theorem 2.4). Whitehead Jr. and Zhao [54] expressed the chromatic polynomial of K_4-homeomorphs as follow.

Lemma 2.4 *(Whitehead Jr. & Zhao [54]) The chromatic polynomial for $G = K_4(a, b, c, d, e, f)$ is*

$P(G, \lambda) =$

$$\left(\frac{1}{\lambda^2}\right)(-1)^m s\Big[s^{m-1} - \left(s^{a+c+f} + s^{c+b+d} + s^{a+b+e} + s^{d+e+f} + s^{a+d} + s^{b+f} + s^{c+e}\right) + (s+1)\left(s^a + s^b + s^c + s^d + s^e + s^f\right) - (s+1)(s+2)\Big],$$

where $s = 1 - \lambda$ and m is the number of edges in G.